Abstract Algebra

Notes missing on the things I read…

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1 Basics

1.1 Set theory

I seem to recall having read somewhere that set theory is the starting point for all mathematics. I don't know about the rest, but it does seem appropriate enough as a foundation for *algebra*…

1.1.1 The empty set

The very special set that contains nothing, usually denoted as $\varnothing$, is contained in all others sets. The usual proof of this is as follows: suppose that there existed a set S that did *not* contain $\varnothing$; this would mean that $\varnothing$ contains at least one element that is not in S—but this contradicts the definition of $\varnothing$ being the *empty* set.

This proof is not intuitionistically valid. An alternative is as follows: given any set A, we always have that for any set B, it holds that $A \setminus B \subseteq A$. Thus if we set $B = A$, we conclude that $A \setminus A = \varnothing \subseteq A$. This is intuitionistically valid, and setting $A = \varnothing$ yields $\varnothing \subseteq \varnothing$.

1.1.2 The powerset

Given a set S, its *powerset* is defined as the set of all of its subsets, and denoted as 2^S. The next result explains the reason for the terminology.

Theorem 1.1. *Given any finite set S, we have $|2^S| = 2^{|S|}$.*

Proof. We use induction on the size of S. It is true when $S = \varnothing$, because $|\varnothing| = 0$ and $2^0 = 1$ (note that $2^\varnothing = \{\varnothing\}$). But exponentiation to zero itself is a convention, so the argument might be more convincing if we start from one: indeed, for any singleton set, $\{a\}$, its powerset has two elements: $\{\varnothing, \{a\}\}$. And accordingly, $2^1 = 2$, so we have a base case.

Now suppose that $S = \{s_1, \ldots, s_n\}$ (we tacitly assume that all the s_i distinct), and that $|2^S| = 2^n$, and let $S' = \{s_1, \ldots, s_n, s_{n+1}\}$. Let T be the set of all the elements of 2^S, plus all the elements of 2^S, with s_{n+1} added to them. That is, $T = 2^S \cup \{x \cup \{s_{n+1}\} \mid x \in 2^S\}$. From the fact that $2^S \subseteq 2^{S'}$, it is immediate that $T \subseteq 2^{S'}$. To show the reverse containment, consider an arbitrary element of $2^{S'}$. If it contains s_{n+1}, it is in T; if not, it is in 2^S—but by construction, this means that it is also in T. Hence, $2^{S'} \subseteq T$—and also $2^{S'} = T$.

Thus we conclude that the elements of $2^{S'}$ consist of all the elements of 2^S, plus those same elements, each joined ($\cup$) with $\{s_{n+1}\}$—hence, $2^{S'}$ has twice the size of 2^S. In particular, $|2^{S'}| = 2|2^S| = 2^{|S|+1} = 2^{|S'|}$. ∎

Remark 1.2. This result can also be proved via Pascal's triangle, and the binomial theorem. Indeed given a set S with n elements, the number of distinct subsets is given by:

$$\sum_{k=0}^{n} \binom{n}{k} \tag{1.1}$$

which the binomial theorem tells us equals 2^n.[1] △

1.1.3 On set difference and complementation

Definition 1.3 (Set difference). *Given two sets A and B, the* **set difference** *of A minus B, denoted $A \setminus B$, is defined as the set $\{ x \in A \mid x \notin B \}$.*

Theorem 1.4. *Given sets A, B and C, we have that $A \setminus B = A \setminus C$ if and only if $A \cap B = A \cap C$.*

Proof. We have that $A \setminus B \cup (A \cap B) = A$ and $A \setminus C \cup (A \cap C) = A$. As in both unions the sets are disjoint, it follows that if $A \setminus B = A \setminus C$ holds, then it must be case that $A \cap B = A \cap C$. And vice-versa. ■

Theorem 1.5. *Given sets A, B and C, the following hold:*

(i) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

(ii) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Proof. If $A = \varnothing$, then (i) and (ii) reduce to $\varnothing = \varnothing \cup \varnothing$ and $\varnothing = \varnothing \cap \varnothing$, which are trivially true—so let $A \neq \varnothing$. If $B = C = \varnothing$, then (i) and (ii) reduce respectively to $A = A \cup A$ and $A = A \cap A$, which are similarly true. So let at exactly one of B, C be different from $\varnothing$. Without loss of generality, let $B = \varnothing$, and $C \neq \varnothing$. Then $B \cap C = \varnothing$, and $B \cup C = C$, and thus:

- (i) reduces to $A = A \cup (A \setminus C)$, which is always true.
- (ii) reduces to $A \setminus C = A \cap (A \setminus C)$, which again is always true.

So let *both* $B \neq \varnothing$ and $C \neq \varnothing$. Then clearly $B \cup C \neq \varnothing$, but it could still happen that $B \cap C = \varnothing$. Then (i) reduces to $A = (A \setminus B) \cup (A \setminus C)$. Clearly, $x \in (A \setminus B) \cup (A \setminus C)$ implies that $x \in A$. Conversely, if $x \in A$, then the only way for it *not* to belong to $(A \setminus B) \cup (A \setminus C)$, would be if it belonged to $B \cap C$—which is against the hypothesis that $B \cap C = \varnothing$. Thus we conclude that $A = (A \setminus B) \cup (A \setminus C)$ holds.

So let A, B and C be all different from the empty set, and furthermore let both $B \cap C$ and $B \cup C$ be also not empty. Proceeding with the demonstration:

- (i) ($\rightarrow$) Let x belong to A but not to $B \cap C$. We have three cases:

  1. $x \in B \wedge x \notin C$, in which case $x \notin A \setminus B$ and $x \in A \setminus C$;
  2. $x \notin B \wedge x \in C$, in which case $x \in A \setminus B$ and $x \notin A \setminus C$;
  3. $x \notin B \wedge x \notin C$, and thus $x \in A \setminus B$ and $x \in A \setminus C$.

  In either way we end up with $x \in (A \setminus B) \cup (A \setminus C)$, thus concluding that $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

  ($\leftarrow$) If $x \in A \setminus B$, then it belongs to $A \setminus (B \cap C)$, for if x does not belong to B, it cannot belong to $B \cap C$. Similarly, $x \in A \setminus C \rightarrow x \in A \setminus (B \cap C)$. Hence, it is obvious that if x belongs to either $A \setminus B$ or $A \setminus C$, it also belongs to $A \setminus (B \cap C)$—i.e. $A \setminus (B \cap C) \supseteq (A \setminus B) \cup (A \setminus C)$.

  Both results now imply that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

[1] See theorem 1.1.3 in the Combinatorics report.

- (ii) (→) Let $x \in A$. If $x \notin B \cup C$, then clearly $x \notin B$—meaning $x \in A \setminus B$—and $x \notin C$, which entails that $x \in A \setminus C$. I.e. $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$.

  (←) Again let $x \in A$. $x \notin B$ and $x \notin C$ imply that x cannot belong to $B \cup C$, and hence $x \in A \setminus (B \cup C)$. I.e. $A \setminus (B \cup C) \supseteq (A \setminus B) \cap (A \setminus C)$.

  Both results now imply that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

■

The same type of proof also gives us the classical results of distributivity:

Theorem 1.6. *The operations of* **conjunction** *(∩) and* **disjunction** *(∪) are* **distributive** *in relation to one another. That is to say, the following properties hold:*

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof.

- (i) (→) If $x \in A$, and $x \in B$, then $x \in A \cap B$, and hence it belongs to the right hand side. Similarly, if $x \in A$, and $x \in C$, then $x \in A \cap C$, and hence it belongs to the right hand side. So the left hand side is contained in the right hand side. (←) If $x \in A \cap B$, then it belongs to the left hand side. Similarly if $x \in A \cap C$. And if both $x \in A \cap B$ and $x \in A \cap C$, then again x belongs to left hand side. Hence, the right hand side is contained in the left hand side.

  The two statements together imply that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

- (ii) (→) If $x \in A$, then it clearly belongs to the right hand side. If $x \notin A$, but $x \in B \cap C$, then x again belongs to the right hand side. Obviously, x also belongs to the right hand side if both conditions hold. This shows that the left hand side is contained in the right hand side. (←) Let x be an element of the right hand side. Then either $x \in A$, or, if $x \notin A$, then it must be that $x \in B$ and $x \in C$. But in both situations, x belongs to the left hand side, showing that the right hand side is contained on the left hand side.

  The two statements together imply that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

■

Definition 1.7. *Given two sets A and B, their* **symmetric difference***, denoted $A \triangle B$, defined as follows: $A \triangle B \stackrel{\text{def}}{=} (A \setminus B) \cup (B \setminus A)$. Equivalently, $A \triangle B \stackrel{\text{def}}{=} (A \cup B) \setminus (A \cap B)$.*

The equivalence can be shown by proving that any element on the set $A \triangle B$ according to one definition must also be in the set according to the other definition, and vice-versa. From this we can see that symmetric difference is commutative. Most of these equalities are proven via the same technique, i.e. showing that any element belonging to the LHS must also belong to the RHS, and vice-versa.

The symmetric difference of sets A and B could have also been defined as the set of all elements x that verify the condition:

$$(x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B) \tag{1.2}$$

Note this disjunction is exclusive. This way of looking at symmetric difference is useful for the next result.

Theorem 1.8. *Symmetric difference is associative.*

Proof. Given sets A, B and C, we want to show that $A \triangle (B \triangle C) = (A \triangle B) \triangle C$. From the LHS of the equation we know that exactly one of the following holds:

$$\begin{cases} x \in A \wedge x \notin (B \triangle C) \\ x \notin A \wedge x \in (B \triangle C) \end{cases}$$

The curly braces represent disjunction, although in this case, the nature of the propositions ensures that both cannot hold simultaneously—i.e. the disjunction is exclusive. We can expand this still further (both equations split into two):

$$\begin{cases} x \in A \wedge x \notin B \wedge x \notin C \\ x \in A \wedge x \in B \wedge x \in C \\ x \notin A \wedge x \in B \wedge x \notin C \\ x \notin A \wedge x \notin B \wedge x \in C \end{cases}$$

The exclusivity property still holds (any two of those four equations cannot hold at the same time). As $\wedge$ distributes over $\vee$, we can group the first and third equations, and the second and fourth, as follows:

$$\begin{cases} \big[(x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\big] \wedge x \notin C \\ \big[(x \in A \wedge x \in B) \vee (x \notin A \wedge x \notin B)\big] \wedge x \in C \end{cases}$$

But from the definition of symmetric difference, this is precisely:

$$\begin{cases} x \in (A \triangle B) \wedge x \notin C \\ x \notin (A \triangle B) \wedge x \in C \end{cases}$$

This is equivalent to $(A \triangle B) \triangle C$, which is what was required to show. ■

Remark 1.9 ($\triangle$ as a group operation). Fun fact: for a given set X, its powerset, together with the operation of symmetric difference (SD), *forms a group*. Indeed the SD of two subsets of X is bound to also be a subset of X, so we have closure. Associativity was just dealt with, and the SD of any set with $\varnothing$ is that set itself, so $\varnothing$ is the identity. Finally, the SD of a set with itself is precisely $\varnothing$, so each element is its own inverse. △

Definition 1.10. *Given a set A, that belong to an ambience space Ω, we define A's **complement** (or it's **negation**) as:*

$$\overline{A} \stackrel{\text{def}}{=} \Omega \setminus A \tag{1.3}$$

Remark 1.11. It is immediate from the above definition that $\overline{\varnothing} = \Omega$, and $\overline{\Omega} = \varnothing$. △

Theorem 1.12. $\overline{\overline{A}} = A$.

Proof. Let Ω be the universe. Then $\overline{\overline{A}} \stackrel{\text{def}}{=} \Omega \setminus (\Omega \setminus A)$. Furthermore, we can observe that *for any set A*, $A \cup (\Omega \setminus A) = \Omega$ and $A \cap (\Omega \setminus A) = \varnothing$. So given any element of Ω it belongs to one and only one of A or $\Omega \setminus A$. Hence all the elements not in $\Omega \setminus A$—i.e. $\Omega \setminus (\Omega \setminus A)$—must be in A. This shows that $\Omega \setminus (\Omega \setminus A) \subseteq A$. For the converse direction, observe that all the elements of A do *not* belong to $\Omega \setminus A$, and hence they belong to $\Omega \setminus (\Omega \setminus A)$, i.e. $A \subseteq \Omega \setminus (\Omega \setminus A)$. Thus $A = \Omega \setminus (\Omega \setminus A)$, and the theorem follows. ■

Theorem 1.13 (de Morgan's laws). *For sets B and C, the following holds:*

(i) $\overline{B \cap C} = \overline{B} \cup \overline{C}$.

(ii) $\overline{B \cup C} = \overline{B} \cap \overline{C}$.

Proof. Follows immediately from the definition of negation (def. 1.10), and from theorem 1.5, setting $A = \Omega$. ■

For the next theorem, an auxiliary result is needed.

Lemma 1.14. *Given sets A, B, we have $A \setminus B = A \cap \overline{B}$.*

Proof. Follows directly from the definitions of set difference and complementation. ■

Theorem 1.15. $\overline{A} \triangle \overline{B} = A \triangle B$.

Proof. $\overline{A} \triangle \overline{B} = (\overline{A} \cup \overline{B}) \setminus (\overline{A} \cap \overline{B}) = \overline{A \cap B} \cap (A \cup B) = (A \cup B) \setminus (A \cap B) = A \triangle B$. ■

1.1.4 Other properties

Theorem 1.16. *Given sets A, B and C, if either $A \subseteq B$ and $B \subset C$, or $A \subset B$ and $B \subseteq C$, then $A \subset C$.*

Proof. In the first case, as $B \subset C$, there is (at least) one element in C that is not in B; and as A is contained in B, then there is also at least one element in C that is not in A—and hence, $A \subset C$.

Similarly for the second case, there an element in B that is not in A; and as B is contained in C, there is also an element in C that is not in A—and so, $A \subset C$. ■

1.2 Induction *a la* Odifreddi

First of all, it more than customary to have induction start from 0—and Odifreddi is no exception. This is for convenience, of course, but we can also start the inductive process from a number superior to 0. Odifreddi, however, while approaching induction with a different formalism from what one usually encounters, also starts from 0: and this is also natural, for he wishes to study computable functions, which are partial computable functions which domain happens to be (the whole of) $\mathbb{N}$. As an exercise, here we rewrite his formalism, but with an arbitrary (positive) starting point. First a bit of notation.

$$(\exists x \leq y)\varphi(x) \stackrel{\text{def}}{=} \exists x\,(x \leq y \wedge \varphi(x)) \tag{1.4}$$

$$(\forall x \leq y)\varphi(x) \stackrel{\text{def}}{=} \forall x\,(x \leq y \to \varphi(x)) \tag{1.5}$$

Note that

$$(\forall x \leq y)\varphi(x) \;\equiv\; \varphi(x) \wedge \varphi(x+1) \wedge \cdots \wedge \varphi(y) \tag{1.6}$$

Working the first two definitions above we get (as we would intuitively expect):

$$\neg(\exists x \leq y)\varphi(x) \equiv (\forall x \leq y)\neg\varphi(x) \tag{1.7}$$

$$\neg(\forall x \leq y)\varphi(x) \equiv (\exists x \leq y)\neg\varphi(x) \tag{1.8}$$

The case for $x \geq y$ is defined similarly (and similar remarks apply to negation).

Definition 1.17 (Axiom of simple induction). *If φ is a formula with one free variable and $r \in \mathbb{N}$ is an arbitrary natural number, then*

$$\Big\{\varphi(r) \wedge (\forall x \geq r)\big[\varphi(x) \to \varphi(\mathcal{S}(x))\big]\Big\} \to (\forall y \geq r)\varphi(y) \tag{1.9}$$

where $\mathcal{S}$ is the successor function.

The meaning of this axiom is perhaps best conveyed if instead of the "everyday" meaning of induction, we instead interpret $a \to b$ as meaning that we cannot have a true and b false. Simple induction is then merely the statement that if for a given proposition φ, it holds for a natural number r, and we can show that it cannot be true for a natural and false for its successor, then we axiomatically believe it holds for all naturals greater or equal than r.

But what if the truth of $\varphi(x+1)$—i.e. φ of the successor of x—depends not on the truth of $\varphi(x)$ alone, but on the truth of proposition φ for a subset of the previous values? Such a subset could even not include x, or on the other extreme, include *all* values smaller than $x+1$. Consider a nonempty set S for which elements the proposition φ holds (the base cases). You might be able show that this implies that it holds for the next value—and this might follow from φ holding just for some of the previous values (i.e. only some of the elements of S). And for the value after that, maybe its truth again follows from φ holding only for *some* of the previous values. And similarly for next value, and so on. In all of these scenarios, when we get to value x, we have already established that φ holds for *all* previous values—even though in each step, we might only use the fact that φ holds some subset of those previous values (and the same might again happen trying to prove that $\varphi(x+1)$ holds).

Hence as a next step we might try to see what happens on the assumption that φ holds for all values up to and including x. If from this it can be deduced that $\varphi(x+1)$ holds, then taking as a proposition the statement that φ holds for all values up to and including x, and applying weak induction on *that* "higher level" proposition, we get the principle of *strong induction*. The simplest way to state it, is to write it like this:

Theorem 1.18. *The* **naive** *form of strong induction:*

$$\varphi(r) \wedge (\forall z \geq r)\Big[\Big\{(\forall x \mid r \leq x \leq z)\varphi(x)\Big\} \to \varphi(z+1)\Big] \to (\forall y \geq r)\varphi(y) \tag{1.10}$$

where the quantifier in the inner implication is defined as would be expected:

$$(\forall x \mid r \leq x \leq z)\varphi(x) \stackrel{\text{def}}{=} \forall x\,(r \leq x \leq z \to \varphi(x)) \tag{1.11}$$

Proof. It follows from simple induction, as hinted above. Let

$$\Phi'(z) = (\forall x \mid r \leq x \leq z)\varphi(x) \tag{1.12}$$

i.e. equal to the antecedent of the inner implication.[2] To prove (1.10), suppose the antecedent of the outer implication (i.e. the inner implication) holds, for all $z \geq r$. This inner implication holds if and only if $\Phi'(z) \to \Phi'(z+1)$ also holds, also for all $z \geq r$ (cf. lemma 1.19).

And the fact that $\Phi'(z) \to \Phi'(z+1)$ holds, together with the fact that $\Phi'(r)$ (i.e. $\varphi(r)$) also holds, allow us to conclude, via simple induction, that $\Phi'(z)$ holds for all $z \geq r$—which can only happen if the same is true of $\varphi(z)$. ■

[2] Equivalently, you can think of (1.12) as $\Phi'(z) = \varphi(r) \wedge \varphi(r+1) \wedge \cdots \wedge \varphi(z)$.

Lemma 1.19. *In the proof of theorem 1.18, we have that* $\Big\{(\forall x \mid r \le x \le z)\varphi(x)\Big\} \to \varphi(z+1)$ *holds if and only if* $\Phi'(z) \to \Phi'(z+1)$ *also holds.*

Proof. First, note that given two implications $a \to b$ and $c \to d$, to check that one holds if and only if the other also holds, it is easier to to check that their *negations* are equivalent. That is, to check that $a \wedge \neg b$ is equivalent to $c \wedge \neg d$. So we have the two implications

$$\textbf{a)}\ (\forall x \mid r \le x \le z)\varphi(x) \to \varphi(z+1) \quad \text{and} \quad \textbf{b)}\ \Phi'(z) \to \Phi'(z+1)$$

$\Phi'(z)$ is equal to the antecedent of a) by definition. And if $\varphi(z+1)$ is false, then clearly $\Phi'(z+1)$ is also false. Conversely, if $\Phi'(z+1)$ is false, *while* $\Phi'(z)$ *is true*, this can only be because $\varphi(z+1)$ is false. ■

But this way of laying out strong induction requires an extraneous condition, which modern mathematical aesthetics denounces as inelegant. So if it can be suppressed… suppress it (of course, this new, more "elegant" form is harder to read, as it requires parsing through the implicit parts, but well, aesthetics seldom comes without a cost…). This is why this new more "aesthetic" form of strong induction prompts a more lengthy discussion afterwords. The superscript asterisks (*) mark the changes in relation to (1.10).

Theorem 1.20 (Strong induction). *For any proposition* φ *with one free variable, the following holds:*

$$(\forall z \ge r)\Big[\Big\{(\forall x \mid r \le x <^* z)\varphi(x)\Big\} \to \varphi(z)^*\Big] \to (\forall y \ge r)\varphi(y) \tag{1.13}$$

Proof. Let $\Phi(z)$ denote the antecedent of the inner implication of (1.13), i.e. $\Phi(z) = (\forall x \mid r \le x < z)\varphi(x)$ (note the difference from Φ', viz. we have here $< z$ rather than $\le z$). Suppose the inner implication of (1.13) holds (for all $z \ge r$). This happens if and only if $\Phi(z) \to \Phi(\mathcal{S}(z))$ also holds (idem.). Indeed—cf. the discussion above, after (1.10)—$\Phi(z)$ is exactly the antecedent of the inner implication of (1.13). And if $\varphi(z)$, the consequent of said implication, is false, then so is $\Phi(\mathcal{S}(z)) = \varphi(r) \wedge \varphi(r+1) \wedge \cdots \wedge \varphi(z)$. Conversely, if $\Phi(\mathcal{S}(z))$ is false, while $\Phi(z)$ is true, that can only mean that $\varphi(z)$ is false.

Now $\Phi(r)$ is (vacuously) true, and thus, applying simple induction to Φ it follows that $(\forall z \ge r)\,\Phi(z)$—which is the same as $(\forall y \ge r)\,\varphi(y)$. ■

Skip base case check?? A cursory look at the above proof might seem to suggest that we can skip checking the base case in certain circumstances, i.e. when it (seems to) follow vacuously—but this is a mistaken view. When $z = r$, the inductive step, $\Phi(z) \to \Phi(\mathcal{S}(z))$, turns into $\Phi(r) \to \Phi(r+1)$, or equivalently, $\Phi(r) \to \varphi(r)$. Now as $\Phi(r)$ is vacuously true, stating that the last implication holds is the same as stating that $\varphi(r)$ holds—but this has to be checked separately; i.e. the "proof of the implication" part one usually does in inductive, er, proofs, will never work starting from the empty set of $\Phi(r)$. Thus, we still always need to check the base case; i.e. to check that there is a nonempty set where Φ (and thus also φ) hold. To make this crystal clear, suppose that the base case $\varphi(r)$ did *not* hold. Then the implication $\Phi(r) \to \varphi(r)$ would be false, meaning that the antecedent in the outer implication of (1.13), with $z = r$, would also be *false*—and so would the consequent: if φ is false for r, it cannot be true for all values y such that $y \ge r$.

Hence, everything still depends on verifying the base case. The difference in relation to (1.10) is that in (1.13), that requirement is hidden under the statement that $\Phi(z) \to \Phi(\mathcal{S}(z))$ holds for all z.[3]

Remark 1.21 (Defining the base case by convention). In [2, §1.1], the fact that *all nonzero* integers can be written (possibly non-uniquely) as a product of primes is proved as follows: if the integer is negative we can multiply by -1, and so we can work only with positive integers. Now by convention we set that a product of zero factors is 1. So 1 is clearly a product of (zero) primes. Now assume that all integers less than n can be written as a product of primes. If n is prime, is is the product of one prime. If it is not prime, we've covered the case $n = 1$, so assume $n \neq 1$. Then it can be written as the product of at least two numbers, each smaller than n; by the induction hypothesis, these can be written as a product of primes—and thus so can n.

If the reader is left with the feeling that this involved a bit of cheating, I sympathise. One is indeed left with the feeling that the induction functions *because* of the convention establishing the base case. Actually however, it is the reverse: the base ($n = 1$) is set—by convention—to such a value that, applying the inductive process to it, leads to the conclusion that the proposition is also true for the next value, $n = 2$, which would be our "more natural" starting point anyway. Indeed, if we start instead with $n = 1$, and move the next case $n = 2$, well 2 is either prime (which is indeed the case), or, if 2 was not a prime, that would mean it could be written as product of 1's, which would in turn mean that 2 was also the result of the product of zero primes. Which is of course absurd, but goes on to show that the inductive reasoning is nonetheless valid: if 1 were a product of zero primes, then 2 would have also to either be prime or be the product of zero primes. And then the same reasoning can be done for $n = 3$, because all numbers before it are products of (possibly zero) primes; and so on…

So in general it seems the pattern is always the same, in these sort of contrived examples: set by convention a starting point that actually leads (via the inductive process) to the more natural starting point—and from there onwards, it's "induction as usual". △

1.2.1 The Well-Ordering Principle

The above discussion shows that weak induction implies strong induction. The converse, which also holds, is usually shown by showing first that strong induction implies the *well-ordering principle* (see below), and then that the WOP implies weak induction (idem.). Note that this means that weak induction, strong induction, and the WOP are all equivalent, in some sense (again, more on this below).

Definition 1.22 (well-ordering principle (WOP)). *Let φ be a proposition, that is true for at least an element x. Then there exists an element y which is the* **smallest** *element for which φ is true.*

Remark 1.23. Given any set S, we can always define a function, sometimes called that set's *characteristic function*, that is true for any element that belongs to S, and false otherwise. Conversely, propositions that are either true or false—or more generally, functions with a binary

[3] For completeness, if there is more than one base case, say $\varphi(r_1)$ and $\varphi(r_2)$, then the antecedent of the outer implication in (1.13) will contain, in particular, the following two conditions: $[(\forall x \in \varnothing)\varphi(x)] \to \varphi(r_1)$ and $\varphi(r_1) \to \varphi(r_2)$. In the first case the consequent clearly does not follow from the antecedent, and in the second, it also cannot follow from the antecedent, otherwise $\varphi(r_2)$ would not be a base case. Hence, when we state that both implications are true, what we are actually saying is that both $\varphi(r_1)$ and $\varphi(r_2)$ need to be checked independently of the inductive reasoning.
Of course, it is syntactically obvious that $[(\forall x \in \varnothing)\varphi(x)] \to \varphi(r_1)$ means that $\varphi(r_1)$ needs to be check separately; but we can draw no such conclusions just from the form of $\varphi(r_1) \to \varphi(r_2)$. Here we must look at the concrete scenario; for example, for the associative property (§ 1.3.3), one of the base cases is $n = 3$, and it is a base case precisely because it does not follow from the previous cases, $n = 1$ and $n = 2$.

output—implicitly define a set consisting of the elements for which they are true (and another one for those for which they are false). Hence, the WOP as stated above is equivalent to saying that every nonempty set has a smallest element. △

We can take an informal shortcut to argue that strong induction implies weak induction. Assuming we have a set of base case(s) already verified, and letting r be the smallest of said cases, strong induction means that if we never observe $\varphi(r), \varphi(r+1), \ldots, \varphi(z)$ being true, and $\varphi(z+1)$ being false (for arbitrary r, z, with $z \geq r$), then we believe that φ is true for all $z \geq r$. Then, making $z = r$, this means we will also never observe $\varphi(z)$ being true and $\varphi(z+1)$ being false (again for an arbitrary $z \geq r$)—and hence we can (informally) say that this latter condition also warrants the belief that φ is true for $z \geq r$. But this is precisely weak induction!

On the meaning of saying that weak and strong induction, and WOP, are equivalent. Before showing that strong induction leads to the WOP, which in turn leads to weak induction, it is worth to pause for a moment and ask: what does it *mean* to say that weak induction implies strong induction? Or that strong induction implies the WOP, or that the WOP implies weak induction? Certainly any proposition that can be proved through weak induction can be also be proven from strong induction, but the converse is clearly false, because we assume more with strong induction. My view is that both weak and strong induction, as well as the WOP, are equivalent in the following sense: if we choose any one of them as an axiom, we can prove the other two forms, and thus "reach" (i.e. prove to be true) the same set of propositions. Thus, in the axiomatic model—where you have to start from somewhere—weak induction, strong induction and the WOP are all *equivalent starting points*.

From strong induction to the WOP. The intuition here is actually quite simple: suppose there is a set of natural numbers that has no smallest element. Then it clearly cannot contain 0, as it is the smallest element of $\mathbb{N}$. But then, it cannot contain 1 either, because then *it* would be the smallest element. And it also cannot contain 2, and so on… Thus, by the principle of (strong) induction, the original set must be empty—and this is the essence of the WOP: every **nonempty** set of naturals must have a smallest element.

Reasoning formally, we need only to take the contrapositive of (1.13):

$$(\exists y \geq r)\neg\varphi(y) \to (\exists z \geq r)[(\forall r \leq x < z)\varphi(x) \wedge \neg\varphi(z)] \tag{1.14}$$

Now write ψ for $\neg\varphi$; as φ is an arbitrary proposition, so is ψ. Thus we get:

$$(\exists y \geq r)\psi(y) \to (\exists z \geq r)[(\forall r \leq x < z)\neg\psi(x) \wedge \psi(z)] \tag{1.15}$$

This says that for any proposition ψ, if it holds for some value greater or equal to r, then there exists (in the same range) a *smallest* value z for which it also holds (which is not necessarily equal to r). Now in practice, when applying induction, it is customary to set r to the smallest element for which the proposition in question holds—the eponymous base case. But this needn't be so: formally, both (1.9) and (1.13)—hence also (1.14) and (1.15)—are *tautologies*, i.e., they are true for an arbitrary r (and indeed, an arbitrary φ). And as we are looking at smallest elements, we can set $r = 0$ and we get precisely the good old WOP: if ψ is true for some natural number y, then there exists a natural z which is the smallest element for which ψ is true.[4] Formally, we get the **well-ordering principle**:

$$(\exists y \geq 0)\psi(y) \to (\exists z \geq 0)[(\forall 0 \leq x < z)\neg\psi(x) \wedge \psi(z)] \tag{1.16}$$

[4] Also note that from the case $r = 0$, follow all others: if $\mathbb{N}$ has a smallest element (0), then so does every subset of $\mathbb{N}$.

Excursus: from weal induction to the WOP?! What would happen if we took the contrapositive to weak induction? The WOP involves a sort of "broad view"—the proposition holds for one element (the minimum) *and for no other below that minimum*—and so does *strong* induction, particularly when starting from 0: if a proposition holds for *all* elements below a given one, then it holds for that one as well. The result of taking the contrapositive of the weak induction principle (cf. (1.9), replacing $\neg\varphi$ with ψ) is coherent with this relation: you get that statement that if a proposition holds for any element greater than r, then either it does *not* hold for the smallest element (r), or there exists some other element x, greater than r, such that the proposition holds for x, but not for its predecessor (instead of it holding for x, and for *none* of its predecessors, as in the WOP). This is a weaker form of the WOP—and in both it and weak induction, the "broad view" from above is replaced with a more "localised" one.

From the WOP to weak induction. The last remaining bit to show to come full-circle—i.e. to show that weak induction, its strong(er) counterpart, and the WOP are equivalent, as discussed above—is to show that the WOP implies weak induction. By way of deriving a contradiction, suppose that the WOP is true, but that weak induction is *false*. That is, that there is a proposition ξ that is true for a natural r, and that holds for $n+1$ whenever it also holds for n, but that is *not* true for all naturals. Let S then be the set of elements for which ξ is false. If such a set is not empty, then it must have a minimal element, according to the WOP. Let m be that element. As the base case is r, we must have $m > r$; but as m is the smallest element for which ξ is false, it must be true for $m-1$. But then, the assumed hypothesis implies that it must also be true for m—a contradiction! Thus the set S is indeed empty, and the principle of weak induction holds.

I end this section noting that induction does not apply only to the integers. In fact, it can be applied to other structures, as long as a special type of ordering relation—called a *well-order*—can be defined. It is the fact that the integers have such an ordering relation that allows the WOP to be meaningfully defined. This is the subject of the next section.

1.3 Binary Relations

Consider a set S; a *binary relation* is a subset of $S \times S$. For $a, b \in S$, if the ordered pair (a, b) belongs to that subset, it is denoted aRb. A relation can be seen as a generalisation of functions. Things get interesting when we impose some structure on those subsets.

Definition 1.24. *A binary relation on a set S is called an* **order relation** *if, for all $a, b, c \in S$, it satisfies the following conditions:*

1. **Reflexivity***: aRa;*
2. **Transitivity***: $aRb \wedge bRc \Rightarrow aRc$;*
3. **Anti-symmetry***: $aRb \wedge bRa \Rightarrow a = b$.*

Note that nothing is being said about *totality*; this will be the topic of §1.3.1. We can, in fact, in an order relation, have two elements a and b such that neither aRb nor bRa holds—in this case the order is said to be a *partial order*. For example, the integers ordered by the divisibility relation, $|$, is a partial order (as we can have two integers such that one is neither a multiple nor a divisor of the other, and vice-versa; e.g. 2 and 5). If any one element is related to every other element, then it is a *total order*.

Given a binary relation, if for two elements a and b, both aRb and bRa hold, this can be seen inducing a notion of "equality" between those elements, in some sense. With an order relation, that equality notion coincides with strict equality (because aRb and bRa can only happen if $a = b$). Thus, we indeed get a notion of *order*: as only one of aRb and bRa happen, then this is an absolute difference, i.e., if $a \neq b$ we can say that one of them is greater than the other.

We can go to the other extreme, and make R coincide with (what we shall mean by) equivalence between two elements. That is, we *loosen* the notion of equality, so that aRb means that (from the "point of view" of R) a is equivalent to b. Note that for this to be meaningful aRb must imply bRa; in indeed we have:

Definition 1.25. *An* **equivalence relation** *on a set S is a binary relation which is reflexive, transitive and* **symmetric***: $\forall a, b \in S \mid aRb \Rightarrow bRa$.*

Observe that with equivalence relations, a notion of order is impossible: indeed, as whenever aRb happens, so does bRa, there is never an absolute difference of the kind described above. Somewhere between these two extremes lie *pre-orders*, in which said absolute difference can exist—i.e. there exist a, b such that $a \neq b$, and only one of aRb or bRa happen. But there can also exist c, d with $c \neq d$, and where both cRd and dRc hold—meaning the binary relation is *not* anti-symmetric:

Definition 1.26. *An order relation which is reflexive and transitive is called a* **pre-order**.

Very informally, pre-orders can be thought of as equivalence relations where the "equivalence classes" are smaller, and there is a loose ordering of such "classes." An order relation is then a degenerate case of a pre-order, where all the "equivalence classes" contain just one element; and an equivalence relation is a pre-order degenerated in the "other direction", i.e. the equivalence class of an element contains all other elements with which it is related.

To show that the hierarchy of "equivalence classes" in a pre-order is independent of the chosen representatives. Suppose that, for a given pre-order, only aRb holds (and bRa does not). Then given any element c such that cRa and aRc hold (i.e. it belongs to the "equivalence class" of a), and given any element d such that dRb and bRd hold ("equivalence class" of b), then we always have that only cRd holds (and not dRc). In words, if a is "smaller" than b, then any element in the "equivalence class" of a is "smaller" than any element in the "equivalence class" of b. This is consistent with the idea of an hierarchy of "equivalence classes," as outlined above.

Indeed, for element c, from transitivity we get that $cRa \wedge aRb \rightarrow cRb$. But bRc cannot hold, otherwise, again from transitivity, we would get $bRc \wedge cRa \rightarrow bRa$, which by hypothesis does not hold. And similarly for element d, from transitivity we get that $aRb \wedge bRd \rightarrow aRd$. But dRa cannot hold, for then we would get $bRd \wedge dRa \rightarrow bRa$, which again is against the hypothesis.

Finally, applying transitivity one last time, we get $cRb \wedge bRd \rightarrow cRd$; and the same conclusion also comes from $cRa \wedge aRd \rightarrow cRd$. But dRc cannot hold, otherwise we would get (for instance) $dRc \wedge cRa \rightarrow dRa$, which we shown above to be impossible. Thus c is "smaller" than d, and as both are arbitrary elements, our hierarchy of "equivalence classes" does not depend on the chosen representative.

Remark 1.27 (To be or not to be (strict)). Binary relations are said to be **strict**, or **irreflexive**, if $\forall x, \neg xRx$. However we will usually assume that R *is reflexive* (unless otherwise noticed), and denote its irreflexive counterpart by R^*. That is aR^*b is used to mean $aRb \wedge a \neq b$, or equivalently, aRb is used to mean $aR^*b \vee a = b$. The same distinction holds for orderings, and henceforth we will use the notation $(S, \preccurlyeq)$ to state that set S is ordered by $\preccurlyeq$, which is not strict

(i.e. it is reflexive). Its strict counterpart is $a \prec b$, which means $a \preccurlyeq b \wedge a \neq b$ (or equivalently, $a \preccurlyeq b$ means $a \prec b \vee a = b$). I use this new notation henceforth when dealing explicitly with orders.[5] △

Note that the negation of reflexivity is *not* irreflexivity. Indeed, we have

$$\neg(\forall a, aRa) = \exists a \mid \neg aRa \tag{1.17}$$

which is very different from the condition of irreflexivity, viz. $\forall a, \neg aRa$. Similarly, the negation of anti-symmetry is not symmetry, nor vice-versa. But if the condition for symmetry "always fails", in the sense that if we know that aRb holds, then we also that bRa does not, then we get the notion of *asymmetry*.

Definition 1.28. *A binary relation R is said to be* **asymmetrical** *if $aRb \rightarrow \neg bRa$, for all a, b.*

In line with what is said above, the negation of asymmetry—$\exists a, b \mid aRb \wedge bRa$—is neither symmetry nor anti-symmetry.

Lemma 1.29. *Asymmetry holds if and only if both irreflexivity and anti-symmetry hold.*

Proof. ($\rightarrow$) If R is asymmetrical it is irreflexive, because the only way the condition $\forall a, aRa \rightarrow \neg aRa$ is true, is if aRa is false for all a, which is exactly irreflexivity.

Furthermore, asymmetry also means the antecedent of the anti-symmetry condition is always false, hence anti-symmetry holds vacuously.

($\leftarrow$) Rewrite anti-symmetry as $a \neq b \rightarrow (\neg aRb \vee \neg bRa)$. From irreflexivity we know that if aRb, then $a \neq b$, and from the new form of anti-symmetry, we conclude that $\neg aRb \vee \neg bRa$ must hold. But as we have assumed aRb, then it must be the case that $\neg bRa$. Hence $aRb \rightarrow \neg bRa$, i.e. asymmetry. ■

An irreflexive relation can be symmetric; in this case the symmetry condition for when $a = b$, $aRa \rightarrow aRa$ will always be true, because both antecedent and consequent will be false. **An irreflexive relation can also be anti-symmetric**; in this case the anti-symmetry condition for when $a = b$, $aRa \wedge aRa \rightarrow a = a$, will be trivially true, because the antecedent will always be false. Also in this case, by lemma 1.29, such a (strict) relation will also be asymmetrical.

[5] For the formalism aficionado, we can redo the reasoning above in a more formal (and arguably more obscure) way. For example to show that, starting from the definition of aR^*b ($\stackrel{\text{def}}{=} aRb \wedge a \neq b$), we get $aRb \equiv aR^*b \vee a = b$, we can do:

$$\begin{aligned} & aR^*b \equiv aRb \wedge a \neq b \\ \Leftrightarrow\ & aR^*b \vee a = b \equiv (aRb \wedge a \neq b) \vee a = b \\ \Leftrightarrow\ & aR^*b \vee a = b \equiv (aRb \vee a = b) \wedge \underbrace{(a \neq b \vee a = b)}_{\text{always} = 1} \\ \Leftrightarrow\ & aR^*b \vee a = b \equiv aRb \vee a = b \end{aligned}$$

Whenever $a = b$ is true, aRb is also true. Hence $aRb \vee a = b$ has the same truth value as aRb—the only way for the truth values to be different, would be if aRb was false and $a = b$ was true, which is impossible (as R is non-strict). And so we conclude that

$$aR^*b \vee a = b \equiv aRb$$

A similar reasoning can be used to show that $aRb \wedge a \neq b \equiv aR^*b$ (starting from the definition of aRb, viz. that $aRb \stackrel{\text{def}}{=} aR^*b \vee a = b$).

If anti-symmetry fails for all pairs, a restricted form of symmetry ensues. To see how, it is convenient to redefine that condition of anti-symmetry, in the following manner:

$$\begin{aligned}
&(aRb \wedge bRa) \Rightarrow a = b \\
\equiv\ &(\neg aRb \vee \neg bRa) \vee a = b \\
\equiv\ &(\neg aRb \vee a = b) \vee \neg bRa \\
\equiv\ &\neg(aRb \wedge a \neq b) \vee \neg bRa \\
\equiv\ &(aRb \wedge a \neq b) \Rightarrow \neg bRa
\end{aligned}$$

Via a similar reasoning to what was done above, if anti-symmetry fails for all pairs, in the sense that the implication $(aRb \wedge a \neq b) \rightarrow bRa$ now becomes true, this constitutes a restricted form of symmetry, viz. symmetry minus reflexivity (whereas regular symmetry can be or reflexive).

Transitivity. We have the following lemma.

Lemma 1.30. *Given a transitive relation, it is irreflexive if and only if it is asymmetrical.*

Proof. ($\rightarrow$) Set $a = c$ in the defining condition of transitivity (cf. definition 1.24); we get $aRb \wedge bRa \Rightarrow aRa$. If the relation is irreflexive, the consequent of this implication is false for all a; hence for it to be true (because the relation is transitive), the antecedent must also always be false. I.e. if, for instance, aRb is true, then bRa must be false. This is precisely the definition of asymmetry.

($\leftarrow$) If a relation is asymmetrical, then lemma 1.29 immediately shows that it must also be irreflexive. ■

We also have the two following results (where R denotes a reflexive relation, and R^* its strict counterpart).

If R^* is transitive, then so is R. As R^* is transitive, this means that for all a, b, c, $aR^*b \wedge bR^*c \rightarrow aR^*c$ holds. As whenever the antecedent is false, the implication is true, the only way to falsify that condition, when moving from R^* to R, is to set $a = c$, to see if we can falsify the consequent.

Remark 1.31 (Transitivity and (ir)reflexivity). If a non-strict relation R is transitive, then R^* needn't also be so: consider for example the relation R consisting of aRb, bRa, aRa and bRb. It is clearly transitive, however its strict counterpart, R^*, is not: $aRb \wedge bRa \rightarrow aRa$, and yet neither aRa or bRb are a part of R^*.[6]

However, if R^* is transitive, so is R. For reductio, suppose R is not transitive; then there must exist a, b and c such that $aRb \wedge bRc \wedge \neg aRc$ holds. Now as R is reflexive, $\neg aRc$ means $a \neq c$. But if $a = b$ holds, then the previous conjunction becomes $aRa \wedge aRc \wedge \neg aRc$, which is always false. Similarly, if we had $b = c$, said conjunction would instead become $aRc \wedge cRc \wedge \neg aRc$, which is also impossible. Hence a, b and c are all different—but this means that $aR^*b \wedge bR^*c \wedge \neg aR^*c$ also holds, meaning that R^* is not transitive, which is a contradiction. △

1.3.1 Totality and wellness

Definition 1.32. *A* **total relation** *is a relation where, for all $a, b \in S$, either aRb or bRa (or both).*

[6] This could be prevented if R was asymmetrical—but then, by lemma 1.30, R would also be irreflexive, and hence it would coincide with R^*.

The property of totality means that any element is related to every other element. It implies reflexivity (the converse is of course false).

We can define the following two other properties for generic binary relations:

Definition 1.33. *A relation is said to be* **trichotomous** *if exactly one of the following holds:* xR^*y*, or* yR^*x*, or* $x = y$*.*

When we say that a non-strict relation R is trichotomous, it is understood that the trichotomy property applies to its strict version R^*.

Lemma 1.34. *Totality and anti-symmetry hold if and only if trichotomy holds.*

Proof. ($\rightarrow$) We reason by cases (regarding totality):

1. xRy and yRx true: by anti-symmetry $x = y$, and (by definition of R^*) both xR^*y and yR^*x are false.
2. xRy true and yRx false: $xRy \vee yRx \iff (xR^*y \vee x = y) \vee (yR^*x \vee x = y)$; if yRx is false that means $x \neq y$ and $\neg yR^*x$, which yields that only xR^*y is true.
3. xRy false and yRx true: identical to previous, concluding that only yR^*x is true.

($\leftarrow$) Conversely, trichotomy implies both totality and anti-symmetry:

1. If only xR^*y is true, then xRy, so totality holds, and anti-symmetry holds because, as xR^*y together with trichotomy implies $\neg yRb$, the antecedent (of the anti-symmetry condition) is false.
2. If only yR^*x is true, then the reasoning is similar.
3. If only $x = y$ is true, then totality holds (both conditions are true), and for anti-symmetry, both antecedent and consequent are true, so the implication is true as well.

■

Trichotomy implies irreflexivity $\forall x, \neg xRx$. Also, strong trichotomous relations obviously cannot be symmetric.

After this brief foray, we delve again into order relations.

Definition 1.35. *A* **total order** *is an order relation where, for all* $a, b \in S$*, either* $a \preccurlyeq b$ *or* $b \preccurlyeq a$ *(or both).*

An order relation which is not total is a *partial order*.

Definition 1.36 (Well-order). *A total order* $\prec$ *with the additional property that for any nonempty subset* A *of* S *there exists* $a \in S$ *such that for all* $x \in S$ *we have:* $a \preccurlyeq x$ *is called a* **well-order** *relation.*

A set together with a well-order relation is said to be a *well-ordered set*. Knuth XXX defines a well-order as a binary relation that, besides the "smallest element" property above, it is also transitive, and trichotomic. This latter condition implies totality (which implies reflexivity) and anti-symmetry—and so, we have all conditions of definition 1.36. For completeness, we restate Knuth's definition.

Definition 1.37 (Well-order a la Knuth). *A* **well-order** *is a relation $\prec$ that is transitive, trichotomic, and such that for any subset A of S there exists $a \in S$ such that for all $x \in S$ we have: $a \preccurlyeq x$.*

Example. We can construct a well-order for the set of *all* integers: let $x \prec y$ be a binary relation such that $|x| < |y| \vee (|x| = |y| \wedge x < 0 < y)$. Let us show that it is indeed a well-order; we use Knuth's definition.

Minimal element: for any integer x, we always have $0 \preccurlyeq x$.

Trichotomy: we want to show that exactly one of $x \prec y$, $y \prec x$ or $x = y$ holds. If $x = y$ it is clear that neither $x \prec y$ or $y \prec x$ can hold.

If $x \prec y$, then from the definition for $\prec$ it is clear that we cannot have $x = y$. Now if $x \prec y$, we have two cases: a) $|x| < |y|$, which entails that $|y| < |x|$ and $|x| = |y|$ are both *false*—which means $y \prec x$ cannot hold; b) $|x| = |y| \wedge x < 0 < y$, which implies that $|y| < |x|$ and $y < 0 < x$ are both *false*—again entailing that $y \prec x$ cannot hold.

If $y \prec x$, a similar reasoning shows that neither $x \prec y$ nor $x = y$ can hold. This establishes trichotomy.

Transitivity: we want to show that if $x \prec y$ and $y \prec z$ hold, then so must $x \prec z$. We must again break this down by cases.

- If $|x| < |y|$ and $|y| < |z|$ hold, then so does $|x| < |z|$ which means $x \prec z$.
- If $|x| < |y|$ and $|y| = |z| \wedge y < 0 < z$ hold, then so does $|x| < |z|$, which gives us $x \prec z$.
- If $|x| = |y| \wedge x < 0 < y$ and $|y| < |z|$ hold, then so does $|x| < |z|$, which gives us $x \prec z$.
- The missing case is when $|x| = |y| \wedge x < 0 < y$ and $|y| = |z| \wedge y < 0 < z$ hold, but this cannot happen, for y cannot be simultaneously greater and smaller than zero.

So the integers ordered by $\prec$ would be $0, -1, 1, -2, 2, -3, 3, \ldots$.

1.3.2 Induction, revisited

Writing condition 1.37 as an implication (the antecedent of which is another implication), and taking the contrapositive, we get the strong induction condition. Which leads to the question of why do we need transitivity and trichotomy, if it seems we get induction from the least element principle alone?

I think what this means is that well-ordering and induction are fundamentally equivalent, *irrespective of the ordering one chooses to use*. **But** the assumption (axiom) that allows us to extrapolate that from conditions for induction (grosso modo, $a_0 \wedge a_i \rightarrow a_{i+1}$), to the conclusion that the proposition in question holds for **all** elements of the domain, *only makes sense if the order relation is a well-order*. This is (somewhat…) easier to see if we picture induction "mechanically", like dominoes falling in sequence: transitivity ensures there are no cycles, and trichotomy ensures that all elements can *eventually* be reached (if trichotomy was false, then there could exist two distinct elements related to each other, but not related to any other; if neither of these is the starting point of induction, then they would be unreachable).

1.3.3 Associativity

Given a binary operation $\cdot$ that is *associative*, the expression $a_1 a_2 \dots a_n$ is always well-defined, regardless of how one chooses to group the terms when evaluating it. This is shown by induction, where the definition of associativity—$a_1(a_2 a_3) = (a_1 a_2)a_3$—together with the previous, trivial cases of one and two terms, serves as the base case.

We begin with an auxiliary result. Consider a specific parenthesis disposition, defined inductively as $\prod_{i=1}^{n} a_i = \left(\prod_{i=1}^{n-1} a_i\right) a_n$. Applied to five terms for example, this would give $(((ab)c)d)e$; i.e. we associate left to right. Note that although associativity is only meaningfully defined for $n \geq 3$, $\prod$ is well-defined also for $n = 1, 2$. However, as stated above, the relevant base case is $n = 3$, which must always be explicitly verified.

Lemma 1.38. *Let m, n be positive integers. We have:*

$$\left(\prod_{i=1}^{m} a_i\right)\left(\prod_{i=m+1}^{m+n} a_i\right) = \left(\prod_{i=1}^{m+n} a_i\right) \tag{1.18}$$

For example, $((ab)c)((de)f) = ((((ab)c)d)e)f$.

Proof. The result is trivial to verify for $n = 1, 2$. For $n = 3$ we have:

$$\begin{aligned}
&\left(\prod_{i=1}^{m} a_i\right)((a_{m+1}a_{m+2})a_{m+3}) \\
=&\left[\left(\prod_{i=1}^{m} a_i\right)(a_{m+1}a_{m+2})\right]a_{m+3} \\
=&\left(\prod_{i=1}^{m+2} a_i\right)a_{m+3} = \left(\prod_{i=1}^{m+3} a_i\right)
\end{aligned}$$

This generalises easily; in fact, taking Eq. 1.18 as the inductive hypothesis, we obtain for $n+1$:

$$\begin{aligned}
&\left(\prod_{i=1}^{m} a_i\right)\left(\prod_{i=m+1}^{m+n+1} a_i\right) \\
=&\left[\left(\prod_{i=1}^{m} a_i\right)\left(\prod_{i=m+1}^{m+n} a_i\right)\right]a_{m+n+1} \\
=&\left(\prod_{i=1}^{m+n+1} a_i\right)
\end{aligned}$$

■

We can now state the general result.

Theorem 1.39 (General associativity rule). *Let n be a positive integer.* **Regardless of parenthesis layout**, *we always have $a_1 \dots a_n = \prod_{i=1}^{n} a_i$, where $\prod$ is as defined above.*

Proof. The assertion is trivial for $n = 1, 2, 3$, and for $n = 4$ it is easily verifiable, but it depends on the assertion holding for *all* previous values of n. This strongly suggests using strong induction. Note that you **cannot** use **only** $n = 1$ or 2 as base cases here; the proof crucially depends on the property of associativity, which as remarked above, is only defined for $n \geq 3$—you have to

check all the first 3 cases; I also checked $n = 4$ above just for good measure. So let us take the inductive step of saying that $a_1 \dots a_n = \prod_{i=1}^{n} a_i$ holds, *for all positive integers up to and equal to n*. For $n+1$, we note that for $a_1 \dots a_n a_{n+1}$, regardless of parenthesis disposition, there always exists $1 \leq k \leq n$ such that $a_1 \dots a_n a_{n+1} = (a_1 \dots a_k)(a_{k+1} \dots a_{n+1})$. Again, the parenthesis layout in both sub-groups of terms can be anything, but because both of them have n or less terms, by the induction hypothesis we get

$$(a_1 \dots a_k)(a_{k+1} \dots a_{n+1}) = \left(\prod_{i=1}^{k} a_i\right)\left(\prod_{i=k+1}^{n+1} a_i\right)$$

and by Lemma 1.38,

$$\left(\prod_{i=1}^{k} a_i\right)\left(\prod_{i=k+1}^{n+1} a_i\right) = \prod_{i=1}^{n+1} a_i$$

Thus, regardless of parenthesis, $a_1 \dots a_{n+1} = \prod_{i=1}^{n+1} a_i$ always holds, which proves the theorem—and also shows that parenthesis are unnecessary, because there is no ambiguity. ■

1.4 Functions

Let $f : A \to B$ be a function. If there exists $g : B \to A$ such that $g \circ f = id_A$, g is called the *left inverse* of f. And if $h : B \to A$ is such that $f \circ h = id_B$, then h is called the *right inverse* of f.

Theorem 1.40. ***(a)*** *f is* **injective** *if and only if it has a left inverse;* ***(b)*** *f is* **surjective** *if and only if it has a right inverse.*

Proof. **(a)**: ($\to$) Assuming f is injective, to show that it has a left inverse we need only to construct it. Let $g : B \to A$ be defined as:

$$g(b) = \begin{cases} a, & \text{if there exists } a \text{ s.t. } f(a) = b \\ a', & \text{otherwise} \end{cases}$$

where a' is a random value of A. Because of the injectivity of f, if the first case happens, a is unique. This directly gives that $g(f(a)) = a$, $\forall a \in A$—i.e. id_A, which means g is the left inverse of f.

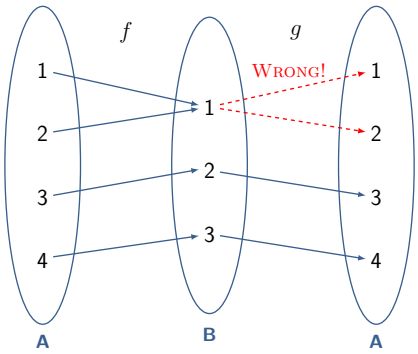
($\leftarrow$) Suppose f has a left inverse g, and that we have $f(x_1) = f(x_2)$. Applying g to both sides yields: $g(f(x_1)) = g(f(x_2)) \iff x_1 = x_2$, which means f has is injective.

(b): ($\to$) If f is surjective, let $h : B \to A$ be such that $h(b) = a$, where a is a (possibly not unique) value such that $f(a) = b$. Because f is surjective, one such value always exists (it is unique if f is also injective). Then clearly $f(h(b)) = b$, i.e. id_B, that is, h is the right inverse of f.

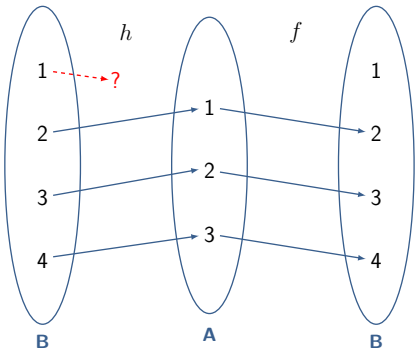
($\leftarrow$) Suppose f has a right inverse $h : B \to A$; then for any element $b \in B$, we have $f(h(b)) = b$. That is, for every element of the codomain of f (i.e. B), there exists an element of the domain of f (i.e. A), namely $h(b)$, which is mapped by f to that codomain element. Hence f is surjective. ■

Remark 1.41. Note that the left inverses (in case **a)**) and right inverses (in case **b)**) defined above *are not unique*. △

Theorem 1.42. *A function $f : A \to B$ is bijective if and only if there there exists $g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$. Furthermore g is also a* **bijection***, and it is* **unique***.*



(a) If f is not injective, it cannot have a left inverse—which sits on the right, when depicted.



(b) If f is not surjective, it cannot have a right inverse—which sits on the left, when depicted.

Figure 1.1: Intuition for theorem 1.40.

Proof. ($\rightarrow$) Construct g as follows. Because f is surjective, for any $b \in B$ there exists $a \in A$ such that $f(a) = b$; and because it is also injective, a is the only such value. Hence $g(b) = a$, where a is the unique value such that $f(a) = b$. The check that $g \circ f = id_A$ and $f \circ g = id_B$ is now routine.

($\leftarrow$) If g in the given conditions exists, then it is both the left and the right inverse of f. Theorem 1.40 then immediately implies that f is injective and surjective—and thus bijective.

For g being bijective, note that in both cases above, f is a left and right inverse of g, and hence g is (also per theorem 1.40) injective and surjective, and thus a bijection.

Now to show that g is unique, let g' be another inverse of f. For all x in the domain of f, both $g(f(x)) = x$ and $g'(f(x)) = x$ must hold. As f is bijective, when x ranges over all the domain of f, $f(x)$ ranges over all the domain of g and g'. Hence, g and g' are equal. ■

Theorem 1.43. *Given $f : A \rightarrow B$ and $g : B \rightarrow C$, we have:*

(i) *f and g injective $\Rightarrow$ $g \circ f$ injective;*

(ii) *f and g surjective $\Rightarrow$ $g \circ f$ surjective;*

(iii) *$g \circ f$ injective $\Rightarrow$ f injective;*

(iv) *$g \circ f$ surjective $\Rightarrow$ g surjective.*

Proof. All the cases are proved via the contrapositive:

(1): If $g \circ f$ is not injective, then there exist x_1 and x_2 such that $x_1 \neq x_2 \wedge g(f(x_1)) = g(f(x_2))$. But this directly implies that either f or g (or both) are not injective.

(2): If $g \circ f$ is not surjective, then either g is not surjective or, f is not surjective (or both).

(3): If f is not injective, then there exist x_1 and x_2 such that $x_1 \neq x_2 \wedge f(x_1) = f(x_2)$, which implies $g(f(x_1)) = g(f(x_2))$, i.e. $g \circ f$ is not injective.

(4): If g is not surjective, then $g \circ f$ cannot possibly be surjective, regardless of the domain of g, or the range of f. ■

Remark 1.44. From (1) and (2) above, we see that the composition of bijections is also bijective. △

2 | Groups

2.1 Groups

A set G with a binary operation that is closed, associative, and has an identity element, and inverses for all elements, forms a group.

An example are the so-called *diehedral groups*, which correspond to "rigid motions" of a regular polyhedron. To better convey what I mean by a rigid motion, imagine a jigsaw piece that happens to be shaped as a regular polyhedron. You remove it, rotate it, flip it, whatever—but in a way that allows you to *put it back in the same place*. How many motions of this kind are there? Well, take any side of the polyhedron, and label one of its vertices A, and the other one B. As the polyhedron has n vertices, for any rigid motion, you have n choices for where you want to place vertex A—but once this is done, you only have *two* available positions for vertex B. Hence, the diehedral group for an n side regular polyhedron, has $2n$ elements.

Weakened axioms. The group axioms for the existence of identity and inverses can be weakened, while still yielding the familiar properties of a group structure. Note that I denote the inverse of an element a as a^{-1}, for reasons that are explained in §2.2; especially circa definition 2.12.

Theorem 2.1. *Let X be a semigroup, i.e. a set with a binary associative operation, where there exists $e \in X$ such that, for all $a \in X$, $ea = a$ holds (that is, there exists a left identity). Furthermore, for all $a \in X$ there exists $a^{-1} \in X$ such that $a^{-1}a = e \in X$. That is, all elements have a left inverse. Then X is a group.*

Proof. We show this by showing first that any left inverse is also a right inverse. We want to come up with an expression where aa^{-1} appears, and which, through associativity, can either evaluate to aa^{-1} or e—thus proving that the left inverse is also a right inverse. $(a^{-1})^{-1}a^{-1}aa^{-1}$ is one such expression:

$$\begin{cases} \left((a^{-1})^{-1}a^{-1}\right)aa^{-1} = aa^{-1} \\ (a^{-1})^{-1}\left(a^{-1}a\right)a^{-1} = (a^{-1})^{-1}a^{-1} = e \end{cases}$$

Now it is only left to show that the left identity is also a right identity. We have $ae = a(a^{-1}a) = (aa^{-1})a = a$, QED. ∎

We could have shown a similar result assuming only right-identity and right-inverses.

Theorem 2.2. *Let G be a group. Its identity element, e, is unique. The same holds for the inverse of any given element.*

Proof. Suppose there was another identity of G, say e'. Then, $ee' = e$ but also $ee' = e'$, so $e = e'$. Now let a' and a'' be two inverses of an element a. We have $(a''a)a' = a'$ but also $a''(aa') = a''$, and so $a' = a''$. ∎

Corollary 2.3. *If two elements of a group have the same inverse, those two elements are equal.*

Proof. Let elements a' and a'' have the same inverse, namely a. We have $(a''a)a' = a'$ but also $a''(aa') = a''$, and so $a' = a''$. ■

Theorem 2.4. $(a^{-1})^{-1} = a$.

Proof. $(a^{-1})^{-1} = (a^{-1})^{-1}e = (a^{-1})^{-1}(a^{-1}a) = ((a^{-1})^{-1}a^{-1})a = a$. ■

We can now prove results like the following (Clark [1], article 26δ):

Theorem 2.5. *Let S be a semigroup with a finite number of elements. If the cancellation laws hold—that is if $ab = ac$ or $ba = ca$, then $a = b$—then S is a group.*

Proof. Let the elements of S be $s_1, \ldots, s_n$, and right-multiply them all by an arbitrary $a \in S$. We obtain $s_1a, \ldots, s_na$. These must all be distinct, otherwise we would have $s_ia = s_ja$ with $i \neq j$, but from the right cancellation law we have $s_i = s_j$, which is contradictory. So the s_ia are all distinct, and as they are in the same number as the elements in the original set, there exists an s_i such that $s_ia = a$, which means have a left identity element—denote it as e. The same reasoning shows that there must exist another element s_j such that $s_ja = e$, i.e. there also exist left inverses.[1] As a is an arbitrary element, this shows that all elements in S have a left inverse. It now follows from theorem 2.1 that S is a group. ■

Remark 2.6. The converse of theorem 2.5—that cancellation laws hold for any group—follows from the group axioms. △

This next result comes from the same place (Clark [1], article 29δ):

Theorem 2.7. *If G is a group such that each element is its own inverse, that is, $x^2 = e$ for all elements, then G is abelian.*

Proof. We have $(ab)(ba) = ab^2a = a^2 = e$, and so, ba is the inverse of ab. As the inverse is unique, and each element is its own inverse, ab and ba must be the same element—hence, the group is abelian. ■

2.2 Exponent Laws

A *monoid* M is a generalisation of a group, where we remove the condition that every element must have an inverse. I.e., it is a set with a binary operation that is closed, associative, and an identity element, e, for which it holds that $ea = ae = a$ for all $a \in M$.[2]

Of course, there can be invertible elements in a monoid—we just remove the requirement that *all* elements need to be invertible (if this happens the monoid is actually a group). If a is an invertible element of monoid M, that means there exists a' such that $a'a = aa' = e$. Note that this implies the inverse is unique, cf. theorem 2.2.

In a multiplicative monoid, exponentiation **to a non-negative power** is defined inductively as follows:

Definition 2.8. *Let a be an element of a monoid M, and $n \geq 0$. Then $a^{n+1} = a \cdot a^n$.*

[1] Note that s_j can be equal to s_i, when a is the identity.

[2] There can be monoids in which one element has (say) two distinct left inverses (in which case said element cannot have a right inverse—why?).

This definition implies that $a^0 = e$;[3] also, due to associativity we have $a^{n+1} = a^n \cdot a$.

The usual exponent laws are valid in any monoid. For *non-negative exponents*, these are:

Theorem 2.9 (Exponent Laws). *Let a and b be elements on a monoid M. Then the following holds:*

(i) $a^n a^m = a^{n+m}$ *for all* $n \geq 0$ *and* $m \geq 0$.

(ii) $(a^n)^m = a^{nm}$ *for all* $n \geq 0$ *and* $m \geq 0$.

(iii) *If* $ab = ba$, *then* $(ab)^n = a^n b^n$ *for all* $n \geq 0$.

Proof. (1) Fix m, and verify that for $n = 0$ the property holds. Now assume it holds for an arbitrary n, and for $n+1$ we obtain $a^{n+1}a^m = aa^n a^m = aa^{n+m} = a^{(n+1)+m}$, where the last equality is due to definition 2.8 (as well as the commutativity and associativity of integer addition).

(2) Fix n, and verify that for $m = 0$ the property holds. Now assume it holds for an arbitrary m, and for $m+1$ we get $(a^n)^{m+1} = (a^n)(a^n)^m = a^n a^{nm} = a^{n(m+1)}$, where the last equality follows from (1).

(3) First prove that $ba^n = a^n b$: it holds for $n = 0$, and if we assume it holds for n, then for $n+1$: $ba^{n+1} = ba^n a = a^n ba = a^n ab = a^{n+1}b$. Now for $(ab)^n = a^n b^n$, it holds for $n = 0$; assuming it holds for n, for $n+1$ comes $(ab)^{n+1} = (ab)(ab)^n = (ab)a^n b^n = baa^n b^n = ba^{n+1}b^n = a^{n+1}bb^n = a^{n+1}b^{n+1}$. ∎

Remark 2.10. Should you feel some unease due to the base being when either n or m is 0, feel free to start at 1—cf. remark 1.21. △

Remark 2.11. In property 3 above, as $ab = ba$, then it must also be that $(ab)^n = (ba)^n$. This means that $a^n b^n = b^n a^n$ also holds. △

Negative exponents. We now come to the main topic in this section, which is how we can generalise these laws to allow *any* integer exponent, including negative ones. Doing this however, first requires that one defines the exponentiation operation for negative powers. To give away the punchline:

negative exponents are only defined for invertible elements—i.e. units!

Let a be an invertible element of a monoid M, and let us represent said inverse as a^{-1}. Then, from the way have defined the inverse, we have that $aa^{-1} = a^{-1}a = a^0 = e$. This is in accordance with integer exponentiation, where given a common base, we add the exponents. This might lead us to consider an element like $(a^{-1})^n$, for some non-negative n. If $n = 0$ we obtain e, but if n is positive, what might the inverse of such an element be? Well, it is straightforward to verify that:

$$(a^{-1})^n a^n = e \tag{2.1}$$

and hence conclude that $(a^n)^{-1} = (a^{-1})^n$. A way of defining exponentiation of negative powers now suggests itself, especially when we take into account the desirability of maintaining that exponent addition property:

Definition 2.12. *Let the a be a unit of a monoid M, and n be a* **positive** *integer. Then* $a^{-n} = (a^{-1})^n = (a^n)^{-1}$.

[3] If we set $n = 0$, then we get $a^{0+1} = aa^0 \iff a = aa^0 \iff a^0 = e$. Note that e is the group identity, sometimes represented by 1, yielding the perhaps more familiar form $a^0 = 1$.

On the other hand, if a is an invertible element with inverse a^{-1}, then a^{-1} is itself invertible:

Theorem 2.13. *$(a^{-1})^{-1} = a$ holds.*

Proof. $(a^{-1})^{-1} = (a^{-1})^{-1}e = (a^{-1})^{-1}a^{-1}a = ea = a$. ∎

This is again in accordance with the usual rule to multiply the exponents. And it allows us to generalise definition 2.12:

Theorem 2.14. *$a^{-n} = (a^{-1})^n = (a^n)^{-1}$ holds for* **any** *integer n.*

Proof. The case for $n = 0$ is obvious, and for $n > 0$ is basically definition 2.12, so let $n < 0$. We have $(a^n)^{-1} = (a^{-(-n)})^{-1}$, and as $-n > 0$, from definition 2.12 comes $(a^{-(-n)})^{-1} = ([a^{-n}]^{-1})^{-1}$. Finally from theorem 2.13 we get $([a^{-n}]^{-1})^{-1} = a^{-n}$. And $(a^{-1})^n = ([a^{-1}]^{-1})^{-n} = a^{-n}$, where the two equalities are again due to same definition and theorem respectively. ∎

Note that this means that the choice of representing the inverse of a as a^{-1} has paid off, because the property of multiplying exponents holds, even when one of those exponents is -1 (and the other is any integer). This in turn allows us to generalise definition 2.8 ($aa^n = a^{n+1}$, for non-negative n):

Lemma 2.15. *For any $n \in \mathbb{Z}$, $a \cdot a^n = a^{n+1}$.*

Proof. If n is non-negative, this coincides with definition 2.8. If n is negative, then

$$\begin{aligned} & aa^n \\ &= a[a^{-1}]^{-n} && \text{(Definition 2.12)} \\ &= aa^{-1}[a^{-1}]^{-n-1} && \text{(Definition 2.8, as } -n \geq 1) \\ &= [a^{-1}]^{-n-1} = a^{n+1} && \text{(Theorem 2.14)} \end{aligned}$$

∎

One useful corollary is the following:

Corollary 2.16. *Given any integer l, another integer $k \geq 0$, and an invertible element a of a monoid M, we can always write $a^l = a^{l-k}a^k$.*

Proof. $a^l = aa^{l-1} = a^2a^{l-2} = \cdots = a^{l-k}a^k$. ∎

Generalisation of exponent laws for arbitrary integers. Now that we have constructed a way of making use of negative exponents, we prove the main result of this section, viz. that the usual exponent laws, proved above for non-negative exponents, also hold for negative ones. The trick turns out to be transforming the negative exponents into positive ones.

Theorem 2.17. *Let a and b be units in a monoid M. Then:*

(i) *$a^n a^m = a^{n+m}$ for all $n, m \in \mathbb{Z}$.*

(ii) *$(a^n)^m = a^{nm}$ for all $n, m \in \mathbb{Z}$.*

(iii) *If $ab = ba$, then $(ab)^n = a^n b^n$ for all $n \in \mathbb{Z}$.*

Proof. The cases when the exponents are both non-negative have been dealt with in theorem 2.9, so we need deal only with the scenarios where either one, or both exponents are negative.

(i) We take first the case when both m and n are negative. We have $a^{m+n} = (a^{-1})^{-(m+n)}$ by theorem 2.14, and this latter expression is equal to $(a^{-1})^{-m+(-n)}$, where both $-m$ and $-n$ are positive. And as such, we apply theorem 2.9 to get $(a^{-1})^{-m}(a^{-1})^{-n}$, from where theorem 2.14 now allows us to write $a^m a^n$.

Now suppose that just one of the exponents is negative; without loss of generality, let it be $n < 0$. We have

$$\begin{aligned} a^m a^n &= a^m (a^{-1})^{-n} && \text{(thm. 2.14)} \\ &= \left(a^{m-(-n)} a^{-n}\right)(a^{-1})^{-n} && \text{(cor. 2.16, as } -n > 0) \\ &= a^{m-(-n)} e = a^{m+n} && (2.2) \end{aligned}$$

(ii) Again first consider the case where both m and n are negative. We have:

$$\begin{aligned} (a^m)^n &= \left\{[(a^{-1})^{-m}]^{-1}\right\}^{-n} && \text{(thm. 2.14)} \\ &= \left\{[(a^{-1})^{-1}]^{-m}\right\}^{-n} && \text{(idem.)} \\ &= (a^{-m})^{-n} && \text{(thm. 2.13)} \\ &= a^{mn} && \text{(thm. 2.9, as both } -m \text{ and } -n \text{ are } > 0) \end{aligned}$$

Now suppose only one of the exponents is negative, say m. Via a similar reasoning, we have $[(a^{-1})^{-m}]^n = (a^{-1})^{-mn} = [(a^{-1})^{-1}]^{mn} = a^{mn}$. If it were $n < 0$, we would have: $[(a^m)^{-n}]^{-1} = (a^{-mn})^{-1} = [(a^{mn})^{-1}]^{-1} = a^{mn}$.

(iii) First note that as $ab = ba$ by hypothesis, so is $a^{-1}b^{-1} = b^{-1}a^{-1}$. Now suppose $n < 0$, as the other case has been taken care of in thm. 2.9. We have:[4]

$$\begin{aligned} (ab)^n &= [(ab)^{-1}]^{-n} && \text{(thm. 2.14)} \\ &= [b^{-1}a^{-1}]^{-n} && \text{(as } abb^{-1}a^{-1} = b^{-1}a^{-1}ab = e) \\ &= [a^{-1}b^{-1}]^{-n} && \text{(above observation)} \\ &= [a^{-1}]^{-n}[b^{-1}]^{-n} && \text{(thm. 2.9, as } -n > 0) \\ &= a^n b^n && \text{(thm. 2.14)} \end{aligned}$$

■

On "additive" exponentiantion. The previous result can also be expressed in additive notation. In this case, a^n is represented as na or an, where a is a monoid element of whatever monoid we are considering, and n is an integer. The inverse of a—which, recall, has to exist for negative exponents to be allowed—is denoted $-a$. So we restate theorem 2.17 as follows:

Theorem 2.18 (Additive exponentiation). *Let a and b be units in an additively denoted monoid M. Then:*

(i) $an + am = a(n+m)$ *for all* $n, m \in \mathbb{Z}$.

(ii) $(an)m = a(nm)$ *for all* $n, m \in \mathbb{Z}$.

(iii) *If* $a + b = b + a$, *then* $(a+b)n = an + bn$ *for all* $n \in \mathbb{Z}$.

[4] The inversion property used in the second step—$(ab)^{-1} = b^{-1}a^{-1}$—is a particular case of a more general property, viz. that $(a_1 \cdots a_n)^{-1} = a_n^{-1} \cdots a_1^{-1}$, provided all the a_i are invertible. It is easily shown by induction.

Remark 2.19. As remarked above, this new notational way of expressing exponentiation is "commutative"; e.g. in item 3 above, $an + bn = na + nb$. △

Also note that there is no problem if the elements of the monoid itself are integers; this is only troublesome for rings (cf. 3.1).

2.3 Subgroups

Definition 2.20 (Subgroup). *Let G be a group. A nonempty subset H of G is a* **subgroup** *if it also verifies the group axioms.*

Theorem 2.21 (Subgroup test). *Let G be a group and H a nonempty subset of G. H is a (sub)group if and only if $a, b \in H \Rightarrow ab^{-1} \in H$.*

Proof. ($\rightarrow$) If H is a (sub)group, the theorem is obvious. ($\leftarrow$) Conversely, if $a, b \in H \Rightarrow ab^{-1} \in H$, then associativity in H follows from the fact that $H \subseteq G$. Now as H is nonempty, it contains at least one element a. Thus $a \in H \Rightarrow aa^{-1} \in H$, i.e. $e \in H$, where e is the identity of G. Furthermore, for any $b \in H$, $eb^{-1} = b^{-1}$ must also be in H, i.e. H is closed for inverses. Finally, given $a, b \in H$, from the previous property we know that b^{-1} is also in H, and thus so is $a(b^{-1})^{-1} = ab$—i.e. H is closed for the group operation of G. ■

If the group is additive, the subgroup test says that it is sufficient to be closed for "subtraction." For multiplicative groups, closeness for "division" suffices.

Consider an arbitrary group G, and any subset S of the set G. There always exists a subgroup of G that contains S. Let H be the smallest of those subgroups (H cannot be smaller than S, so there must exist a smallest subgroup). Then H is precisely the set of the elements of the form $s_1 s_2 \dots s_m$, for any non-negative integer m (having $m = 0$ yields the identity) and with $s_i \in S$ or $s_i^{-1} \in S$. Indeed, it is easy to check that the set $\{ s_1 s_2 \dots s_m \mid s_i \in S \text{ or } s_i^{-1} \in S \text{ and } m \geq 0 \}$ is a subgroup of G. Conversely, suppose there exists a subgroup H' that also contains S, but such that $|H'| < |H|$. Then there exists an element of the form $s_1 s_2 \dots s_m$, which does *not* belong to H'—but as H' contains S, this violates the closure property of groups, and thus H' cannot be a group, which is a contradiction.

Theorem 2.22. *If G is an abelian group, an m is an integer, the set $G^m \stackrel{\text{def}}{=} \{ a^m \mid a \in G \}$ is a subgroup of G.*

Proof. G^m is nonempty, as it contains at least $e^m = e$. Now, given $a, b \in G$ (and thus $a^m, b^m \in G^m$), via the subgroup test, we need to show that $a^m (b^m)^{-1}$ is also in G^m. We have $a^m (b^m)^{-1} = a^m (b^{-1})^m = (ab^{-1})^m$, where this last equality holds because G is abelian. As $ab^{-1} \in G$, $(ab^{-1})^m \in G^m$, and the result is proved. ■

If G used additive notation, we would express the subgroup as mG. An example of this are the subgroups of the integers of the form $m\mathbb{Z}$.[5]

Theorem 2.23. *If G is an abelian group, the set $G\{m\} \stackrel{\text{def}}{=} \{ a \in G \mid a^m = e \}$ is a subgroup.*

Proof. The set $G\{m\}$ is nonempty: $e^m = e$. By the subgroup test, we need to show that if $a^m = e$, and $b^m = e$, then $(ab^{-1})^m = e$. We have $(ab^{-1})^m = a^m (b^{-1})^m = e(b^m)^{-1} = e^{-1} = e$, where the first equality is due to G being abelian. ■

[5] Actually, all integer subgroups are of this form XXX.

2.4 Cosets and Normal/Factor (Sub)groups

Given a group G, a subgroup H, and an element $g \in G$, the set $gH \stackrel{\text{def}}{=} \{ gh \mid h \in H \}$ is called a **(left) coset** of H, with representative g.[6] The corresponding right coset would be Hg, defined as you would expect.

Given $g_1, g_2 \in G$, saying that $g_1H = g_2H$ means that for any $h \in H$, we can always:

- find $h' \in H$ such that $g_1h = g_2h'$;
- find $h'' \in H$ such that $g_2h = g_1h''$.

Theorem 2.24. *Let G be a group, H one of its subgroups, and g_1, g_2 two of G's elements. Then the following are equivalent:*

(i) $g_1H = g_2H$;
(ii) $Hg_2^{-1} = Hg_1^{-1}$;
(iii) $g_1H \subset g_2H$;
(iv) $g_1 \in g_2H$;
(v) $g_2^{-1}g_1 \in H$.

Proof. We shall prove $1 \leftrightarrow 2$, and then $1 \to 3 \to 4 \to 5 \to 1$.

($1 \leftrightarrow 2$) For any $h \in H$, we have from 1 that there exists $h' \in H$ such that $g_1h = g_2h'$ (and vice-versa). Taking the inverse of both sides, we obtain:

$$(g_1h)^{-1} = (g_2h')^{-1} \Leftrightarrow h^{-1}g_1^{-1} = h'^{-1}g_2^{-1} \tag{2.3}$$

As one of h, h' was arbitrary, the same holds for their inverses, which yields equation 2.

($1 \to 3$) Equation 3 is a direct consequence of equation 1.

($3 \to 4$) The identity of G, denote it e, is in H, and so it is also in g_1H. This means that $g_1 \subset g_1H$—and from 3, via transitivity, it must also be that $g_1 \subset g_2H$.

($4 \to 5$) For some $h \in H$, it holds that $g_1 = g_2h \Leftrightarrow h = g_2^{-1}g_1$.

($5 \to 1$) First let $h \in H$. We want to find $h' \in H$ such that $g_1h = g_2h'$. Rewrite this as $g_2^{-1}g_1h = h'$; as we know, from 5, that $g_2^{-1}g_1 \in H$, from the closure of H follows that h' is also in H.

Conversely, given $h' \in H$, we want to find $h \in H$ such that $g_1h = g_2h'$. Rewriting this as $h = g_1^{-1}g_2h'$, and noticing that from 5, we also know that $g_1^{-1}g_2 \in H$—due to H being closed to inverses, and $(g_2^{-1}g_1)^{-1} = g_1^{-1}(g_2^{-1})^{-1} = g_1^{-1}g_2$—and hence, $h \in H$. Thus, 1 holds. ■

Remark 2.25. As H is a subgroup, property 5 is equivalent to having $g_1^{-1}g_2 \in H$. △

Definition 2.26. *Given a group G, a subgroup N is said to be* **normal** *in G if for all $g \in G$, $gN = Ng$.*

Remark 2.27. Note that if G is abelian, all its subgroups are normal. △

Theorem 2.28. *If N is a normal subgroup of G, then we have $gNg^{-1} = N$, for any $g \in G$.*

Proof. First, we need to show that, given any $g \in G$, any $n \in N$ can be written as $gn'g^{-1}$, for some $n' \in N$. As N is a normal subgroup, given any $n \in N$, there always exists another $n' \in N$ such that $ng = gn'$—but this is the same as having $n = gn'g^{-1}$. Second, we need to show the converse, viz., that given $g \in G$, for any $n \in N$, the element gng^{-1} belongs to N. Via the same reasoning, we can, given g, and for any n, write $gn = n'g$, for some $n' \in N$. But then, $n' = gng^{-1}$ is in N. Hence, $gNg^{-1} = N$ holds. ■

[6] Note that if $g \in H$, gH is a permutation of H.

Theorem 2.29 (Quotient group). *Given a group G, with a normal subgroup N, the set of left cosets of N, forms a group, under the operation $(aN)(bN) = (ab)N$. This group is called the* **factor group**, *or* **quotient group**.

Proof. N acts as the identity, and the inverse of aN is $a^{-1}N$. The operation is associative, because:

$$[(aN)(bN)](cN) = (abN)(cN) = ((ab)c)N = (a(cb))N \tag{2.4}$$
$$= (aN)[(cb)N] = (aN)[(bN)(cN)] \tag{2.5}$$

And it is well-defined, because if $aN = a'N$, and $bN = b'N$, then $a' = an_1N$ and $b' = bn_2N$, for some $n_1, n_2 \in N$. So:

$$(a'N)(b'N) = (an_1N)(bn_2N) = (aN)(bN) = (ab)N \tag{2.6}$$

because n_1N and n_2N are just permutations of N, due to the fact that $n_1, n_2 \in N$. ■

The previous result uses multiplicative notation, but we can illustrate the same principle using additive notation, with the group $\mathbb{Z}_n$, of integer addition modulo n.[7] The elements of this group are the equivalence classes, modulo n, of the integers $0, 1, \ldots, n-1$. But this group is also denoted as $\mathbb{Z}/n\mathbb{Z}$. Why? Because $\mathbb{Z}$ is an abelian additive group, and $n\mathbb{Z}$ is a normal subgroup. Furthermore, the cosets $0 + n\mathbb{Z}, 1 + n\mathbb{Z}, \ldots, (n-1) + n\mathbb{Z}$ correspond exactly to the equivalence classes of $0, 1, \ldots, n-1$. And as we can always push extraneous n multiples into $n\mathbb{Z}$, coset addition in the quotient group corresponds to modular addition.

But note that this does NOT happen with $\mathbb{Z}_n^*$! Indeed, as $\mathbb{Z}$ is not a group under multiplication (only 1 and -1 have inverses), $n\mathbb{Z}$ cannot be a subgroup.

[7] See the Number Theory wiki, section "Congruences", for more information on this group.

3 | Rings

3.1 Rings

A *ring* is an algebraic structure $(A, +, \cdot)$ such that $(A, +)$ is an abelian group, and $\cdot$ is associative and distributes over addition. If A contains an identity for the operation $\cdot$, it is called a *ring with unity*. ≪ tabSbS

Notations for $+$, $\cdot$, 0 and 1. Carrying over the similarity with the primordial ring, that of the integers $\mathbb{Z}$, we denote $+$ as addition and $\cdot$ as multiplication—although for a particular ring those operations could actually be something completely different. Moreover, for the same reason, the additive identity (which must always exist) is denoted by 0, and the (optional) multiplicative one, if it exists, is denoted by 1. (This is a significant departure from the case of groups, where the identity was usually denoted by e.) The additive inverse of an element a, is denoted $-a$. Obviously, these operations and elements might have nothing to do with their integer counterparts, but it significantly eases the notational burden.

The multiplicative identity, if it exists, is called an **unity**. Elements for which there exists a multiplicative inverse, if any, are called **units**.

Two types of multiplication. Speaking of notational burden, multiplication between elements of a ring, say between a and b, is often denoted by simply juxtaposing them, ab. This can cause confusion when want to represent the repeated addition of a, say, n times, which is sometimes done as na. To distinguish these two very different operations, the *latter one,* **repeated addition** of a, n times, will be represented as $n \cdot a$ (or $a \cdot n$). For example, we will write $a + a + a$ as $3 \cdot a$, or $a \cdot 3$. From this way of defining things, stems the next result.

Theorem 3.1. *Let a, b be elements of a ring R, and $s, t \in \mathbb{Z}$. We have: $(a \cdot s)(b \cdot t) = (st) \cdot (ab) = (ab) \cdot (st)$.*

Proof.

$$(a \cdot s)(b \cdot t) = \left(\sum_{i=1}^{s} a \right) \left(\sum_{j=1}^{t} b \right) \tag{3.1}$$

$$= \sum_{\substack{1 \le i \le s \\ 1 \le j \le t}} ab = (st) \cdot (ab) = (ab) \cdot (st) \tag{3.2}$$

■

Theorem 3.2. *For any ring, any element a multiplied by the* **additive** *identity (i.e. zero) equals that same identity.*

Proof. We have $0a = (0+0)a = 0a+0a$. Adding to both sides the additive inverse of $0a$, gives $0 = 0a$. For $a0$ the reasoning is analogous. ■

Theorem 3.3. *Given elements a, b of a ring R, we have: $(-a)b = a(-b) = -ab$.*

Proof. We have $(-a)b = (-a)b + ab - ab = (-a+a)b - ab = 0b - ab = -ab$. For $a(-b)$ the proof is similar. ■

Corollary 3.4. *Given elements a, b of a ring R, we have: $(-a)(-b) = ab$.*

Proof. $(-a)(-b) + a(-b) = (-a+a)(-b) = 0b = 0$, so $a(-b)$ is the additive inverse of $(-a)(-b)$. But by the previous theorem (3.3), $a(-b) = -ab$, and $-ab$ is the additive symmetric of ab. Hence $(-a)(-b)$ and ab have the same additive inverse, and so we must have $(-a)(-b) = ab$ (cf. corollary 2.3). ■

Corollary 3.5. *In a ring, the following holds: $a(b-c) = ab - ac$ and $(b-c)a = ba - bc$.*

Proof. $a(b-c) = a(b+(-c)) = ab + a(-c)$, which by theorem 3.3 equals $ab - ac$. For $(b-c)a$ the reasoning is similar. ■

Theorem 3.6. *For any ring with unity, the following hold:*

(i) $(-1)a = -a$.

(ii) $(-1)(-1) = 1$.

Proof. Note that -1 refers to the additive inverse of the multiplicative identity. For 1), $a + (-1)a = 1a + (-1)a = (1 + (-1))a = 0a = 0$, and hence $(-1)a$ is the additive symmetric of a, which we denote as $-a$.

For 2), set $a = -1$ in 1), and recall that $-(-a) = a$.[1] Another way would be to set $a = b = -1$ in corollary 3.4. ■

3.2 Subrings

Unlike what happens with groups, subrings are not really that important—the more important notion turns out to be that of an *ideal*. But they are needed, and they are defined in the obvious way: given a ring R, a subset S of R is a subring if the axioms of rings also hold true for it. Of course, *whatever axioms we use for ring (e.g. with or without unity)* ***must also be used for the subring(s)!***

The following result applies to any ring, with or without unity.

Theorem 3.7. *Let $(R, +, \cdot)$ be a ring. A nonempty subset S of R is a* **subring** *if it is closed for $\cdot$ and $-$ (additive symmetric).*[2]

Proof. ($\rightarrow$) If S is a subring, then it closed for $\cdot$, and as it is also an additive (sub)group, it is also closed for $-$ (cf. the subgroup test). ($\leftarrow$) Conversely, the requirement that S be closed for $-$ implies, also via the subgroup test, that S will be an additive abelian (sub)group. Thus S is closed for $+$, and this, together with the assumption that it is closed for $\cdot$, shows that the

[1] Cf. theorem 2.13, which in additive notation is written precisely as $-(-a) = a$.

[2] The reason why we cannot just require instead that S be closed for $+$ and $\cdot$ is that this is not sufficient to guarantee that S be an (additive) abelian group. Case in point: $\mathbb{Z}^+$ is closed for addition and multiplication, but is not an abelian group (it does not contain additive inverses).

distributivity laws hold for S (as it is a subset of R).[3] Associativity of $\cdot$ holds for the same reason. Hence S is a (sub)ring. ■

This is a very general definition, in particular because it doesn't require that R be a ring with unity (multiplicative identity). We could adapt the result for rings with unity, but that is a somewhat perilous course, because it is possible for a ring with unity, to have a subring also with unity—albeit a *different* unity:

Example 3.8. Consider the ring of integers modulo 6, $\mathbb{Z}_6$, and its subring $\{0, 2, 4\}$. The unity of $\mathbb{Z}_6$ is 1, but the unity of the subring is 4! ◊

3.3 Integral Domains

A specialisation of a ring, that sits between rings and fields, are *integral domains*.

Definition 3.9. *A commutative ring R with unity is an* **integral domain** *if it does* **not** *have any* **zero divisors***; i.e. if given any nonzero element $a \in R$, there is no nonzero element $b \in R$ such that $ab = 0$.*

For integral domains as defined above, the cancellation law holds: if $a \neq 0$, then $ab = ac \to b = c$. However we could have also defined integral domains as rings for which the cancellation law holds, as for such rings, there can be no zero divisors—so the two definitions are equivalent. This is shown in the next two results (hereinafter, when referring to integral domains, that means definition 3.9: rings without zero divisors).

Theorem 3.10 (Cancellation law). *Let a, b, c be elements of a ring without zero divisors (i.e., an integral domain), with $a \neq 0$. Then if $ab = ac$, then $b = c$.*

Proof. $ab = ac \iff a(b-c) = 0$, and $a \neq 0$, so due to nonexistence of zero divisors, we must have $b - c = 0$, or equivalently, $b = c$. ■

Theorem 3.11. *Any ring for which the cancellation law holds, has no zero divisors—i.e., it is an integral domain.*

Proof. I will show the contrapositive, viz. that if a ring contains zero divisors, the cancellation law does *not* hold. Indeed, let a, b be two nonzero elements of a given ring, such that $ab = 0$. Then $ab = a0$, but $b \neq 0$—i.e., the cancellation law does not hold. ■

Remark 3.12. Theorem 3.11 could also have shown like this (still using the contrapositive): suppose that in one ring where cancellation holds, we have nonzero elements a, b, such that $ab = 0$ (i.e. we have zero divisors). Take any element c, and write b as $(b + c) - c$. If we set $d = b + c$, we obtain $b = d - c$, and thus we can now rewrite the first condition as $a(d - c) = 0$, which is equivalent to $ad = ac$—and via cancellation, this would imply that $d = c$, which is false, as $b \neq 0$ (and thus $d \neq c$). Hence cancellation law does not hold. △

A *field* is a ring which is also an abelian group for multiplication—i.e. where multiplication is commutative, and all elements except 0 have a multiplicative inverse. Obviously, all fields are rings—indeed, all fields are integral domains—they just have additional structure. We have the following result:

[3] In more detail, let a, b and c be elements of S. Then $a(b + c)$ is in S due to additive and multiplicative closure. And $ab + ac$ is also in S, due to multiplicative and additive closure. Similar reasoning applies to $(b + c)a$ and $ba + ca$. As the binary operation of S coincides with that of R, which must be well-defined—and as left and right distributivity hold in R—we conclude that the distributivity laws hold in S.

Theorem 3.13. *Any finite integral domain is a field.*

Proof. Take a nonzero element a. We must show that a has a multiplicative inverse. If $a = 1$ we are done, so assume $a \neq 1$. As the integral domain is finite, if we take the sequence of powers of a, $a^2, a^3, \ldots$, eventually we will hit repeated elements. Hence, there must exist positive integers i, j, with $i > j$ such that $a^i = a^j$. Cancelling, we obtain:

$$a^i = a^j \iff a^{i-j}a^j = 1a^j \iff a^{i-j} = 1 \tag{3.3}$$

If $i - j = 1$, that would mean that $a = 1$, which is not the case—so it must be that $i - j > 1 \iff i - j - 1 > 0$. And so $1 = a^{i-j} = a^{i-j-1}a$, meaning that a^{i-j-1} is the inverse of a. ■

3.4 Characteristic of a Ring

Definition 3.14. *Let R be a ring with unity. Its* **characteristic** *n is the least positive integer n such that $n \cdot 1 = 0$.*[4] *If no such integer exists, then the characteristic is 0.*

If a ring has characteristic n, then adding n times any of its nonzero elements, also yields 0: indeed let $x \neq 1$ be one such nonzero element. We have:

$$n \cdot x = n \cdot 1x = \underbrace{(1x + 1x + \cdots + 1x)}_{n \text{ times}} = \underbrace{(1 + 1 + \cdots + 1)}_{n \text{ times}}x = (n \cdot 1)x = 0x = 0 \tag{3.4}$$

For this reason, the **characteristic for a ring *without unity*** is defined as the smallest number of times we need to add *any* one of its elements to itself, in order to obtain 0.

Theorem 3.15. *The characteristic of an integral domain, is either 0 or a prime.*

Proof. We need to show that if the characteristic of an integral domain is positive, then it is prime. So let $n = st$ be the positive characteristic of an integral domain (with s, t, and of course n, positive integers). We have $0 = n \cdot 1 = st \cdot 1 = (s \cdot 1)(t \cdot 1)$, were the last equality is due to property 3.1. As we are dealing with an integral domain, one of $(s \cdot 1)$ or $(t \cdot 1)$ must be 0—but n is by hypothesis, the least positive integer such that $n \cdot 1 = 0$. Hence one of s or t must be equal to n, and the other is equal to 1 (the integer, not the unit of the integral domain). This entails the primality of n. ■

[4] **Beware:** both the 0 and the 1 here refer to the additive and multiplicative identities, respectively! But in the next sentence, having characteristic 0, this 0 is an integer! Cf. the remarks at the beginning of the chapter about the two types of multiplications we have in rings (§3).

4 Equations

The whole point of algebra, originally at least, was to solve equations…

4.1 Quadratic Equations

So we have an equation (in the reals) of the form $ax^2 + bx + c = 0$, and we want to solve it. Usually this requires factorising the expression. But how?

We could start by noting that $(x+d)^2 = x^2+2xd+d^2$. Since we want to obtain bx instead of $2dx$, we might set $b = 2d$, i.e. $d = b/2$:

$$(x + b/2)^2 = x^2 + bx + b^2/4 \tag{4.1}$$

This is closer to the goal, but what we want next is to have c instead of $b^2/4$:

$$x^2 + bx + c = 0 \iff x^2 + bx + b^2/4 + (c - b^2/4) = 0 \tag{4.2}$$

$$\iff (x + b/2)^2 = \frac{b^2 - 4c}{4} \iff x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} \tag{4.3}$$

This is already pretty close. But suppose that in (4.1), we replace $b/2$ with $b/(2a)$. We obtain

$$\left(x + \frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \tag{4.4}$$

which, if we multiply by a, yields:

$$a\left(x + \frac{b}{2a}\right)^2 = ax^2 + bx + \frac{b^2}{4a} \tag{4.5}$$

Our problem is now solved:

$$ax^2 + bx + c = 0 \iff ax^2 + bx + \frac{b^2}{4a} + \left(c - \frac{b^2}{4a}\right) = 0 \tag{4.6}$$

$$\iff a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a} \iff \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{4.7}$$

$$\iff x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2|a|} \tag{4.8}$$

So if $a > 0$ (if $a = 0$ this is a degree 1 equation, and formula is inapplicable) we obtain:

$$\iff x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a} \tag{4.9}$$

If $a < 0$, then:

$$\iff x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2(-a)} = -\frac{b \mp \sqrt{b^2 - 4ac}}{2a} \tag{4.10}$$

But this means x takes exactly the same values as with formula (4.9)—which we take as the formula for the roots of second degree equations.

References

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2. **Shoup**, Victor (2008). *A Computational Introduction to Number Theory and Algebra*, Second edition. Ebook version: Cambridge University Press. Cited on p. 9.