Exponentiation in $\mathbb{R}$: from integer to real exponents

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Abstract. The present text provides the motivations (and/or proofs) for the rules of the operation of raising a real number to integer, rational, and real powers (exponents). We take for granted the usual properties of real numbers, such as existence and uniqueness of additive and multiplicative identities and inverses (multiplicative inverses for nonzero elements only), etc.

1 Introduction

Integers exponents are taken care of in §(2), rational ones in §3, and reals in §4. For the last one, knowledge of sequences, and in particular about computing their limits, is assumed. In the remainder of this section, we deal with some auxiliary topics that, are either put here for completeness—which is the case of the next (named) paragraph—or that will be needed in what is to follow.

The *sui generis* case of 0^0. In what follows, the basis of the exponentiation will always required to be *nonzero*. Moreover, we will define below that, for nonzero a, $a^0 = 1$. However, it is clear that for any *positive integer* x, we must have $0^x = 0$. We will also show below that if x is a negative integer, then 0^x should be the multiplicative inverse of 0^{-x}, which is not defined, as $0^{-x} = 0$. But exponentiation to a rational exponent will be defined based on exponentiation to an integer power, and similarly exponentiation to a real exponent will be defined based on exponentiation to a rational power. Hence, it will turn out that it is also nonsensical to talk about raising 0 to a negative rational or real power—but if x is either a positive rational, or a positive real, we will have $0^x = 0$. Which leaves out only the case of $x = 0$, i.e., of 0^0.

The usual convention is to define $0^0 \stackrel{\text{def}}{=} 1$, because this turns out to massively convenient. For example, consider the following application of

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the binomial theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{1.1}$$

This holds for $x = 0$, if and only if $0^0 = 1$.

Important caveat: The above considerations **do not apply to limits!** For example, if we have a function f such that $\lim_{x\to a} f(x) = 0$ and another function g such that $\lim_{x\to a} g(x) = 0$, we **CANNOT** conclude that $\lim_{x\to a} f(x)^{g(x)} = 1$!! In this case the result is indeterminate. For more details, see the relevant Wikipedia page.[1]

Well-defined operations. In what follows, we will at times speak of operations being *well-defined*. What this means is that the same operation, on the same operands, must produce always the same result. In the case of fractions, it also means that the result of a given operation must be the same, regardless of whatever fraction one chooses to represent a particular rational number.

Applying integer exponentiation to (and taking roots of) both sides of an equality. Let α, β be arbitrary real numbers. Then the following holds:

- **Exponentiation.** If $\alpha = \beta$, then for any n we have $\alpha^n = \beta^n$, due to exponentiation being a well-defined operation. If n is *odd*, the converse is also valid: $\alpha^n = \beta^n$ implies $\alpha = \beta$. If this were *not* the case, then letting $c = \alpha^n = \beta^n$, we would conclude that the equation $x^n = c$ with an odd n has at least two different solutions, which is not true in $\mathbb{R}$. If n is even, then from $\alpha^n = \beta^n$ we can only conclude that $\alpha = \pm\beta$.[2]
- **Roots.** If $\alpha = \beta$, then $\sqrt[n]{\alpha} = \sqrt[n]{\beta}$, for any positive n, as $\sqrt[n]{\ }$ is a well-defined operation. The converse also holds, for any n: $\sqrt[n]{\alpha} = \sqrt[n]{\beta}$ implies $\alpha = \beta$. This follows from the previous exponentiation property: $\sqrt[n]{\alpha} = \sqrt[n]{\beta} \Rightarrow (\sqrt[n]{\alpha})^n = (\sqrt[n]{\beta})^n \Leftrightarrow \alpha = \beta$.

From here, it easily follows that for equations, taking roots of both hand sides is always fine, as is raising both sides to an odd exponent. But raising both sides to an even exponent, may introduce additional solutions: the prototypical example is the equation $x = 2$, which has only one solution, versus $x^2 = 4$, which has *two*—namely, 2 and -2. However, care is still needed when taking roots of both sides; for example, we have $x^2 = 4 \Leftrightarrow \sqrt{x^2} = \sqrt{4}$, but this is **not** equivalent to $x = 2$! The correct simplified form is $|x| = 2$. Which brings us to our next topic…

Simplifying roots and powers. These are immediate consequences the preceding properties. For any real number c, and any positive integer

n, if $\sqrt[n]{c}$ is defined, we *always* have $(\sqrt[n]{c})^n = c$—we might *also* have $(-\sqrt[n]{c})^n = c$, if n is even and c positive, but $(\sqrt[n]{c})^n = c$ always holds.

As for $\sqrt[n]{c^n}$, we have for an odd n, $\sqrt[n]{c^n} = c$. But if n is even, then we can only conclude that $\sqrt[n]{c^n} = |c|$. Both equalities again hold for any real c.

2 Integer Exponents

Let a be a nonzero real number, and m, n be positive integers. We define

$$a^m \stackrel{\text{def}}{=} \underbrace{a \cdot a \cdot a \cdots a}_{m \text{ times}} \tag{2.1}$$

From the associativity of multiplication, it is straightforward to observe the **exponent sum rule**, viz. $a^m a^n = a^{m+n}$. It now follows easily that we also have the **exponent multiplication rule**:

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdot a^m \cdots a^m}_{n \text{ times}} = a^{mn} \tag{2.2}$$

The only missing property is the **basis multiplication rule**, $(ab)^m = a^m b^m$, where b is a nonzero real number, just as a (and m continues to be a positive integer). Unlike the properties above, this one depends crucially on the fact that multiplication is *commutative*. This property is shown by induction—and as the case when the exponent is 1 is trivial, I shall start with $m = 2$: $(ab)^2 = (ab)(ab) = aabb = a^2b^2$. One can already see why commutativity is crucial. Assuming the property holds for an arbitrary m, we have: $(ab)^{m+1} = (ab)^m(ab) = a^m b^m ab = a^m a b^m b = a^{m+1} b^{m+1}$—which establishes the result for positive exponents.

Remark 2.3 (Associativity of the exponents). From (2.2) it also follows that $(a^m)^n = (a^n)^m$. Also, when there are three or ore exponents, we can multiply them **in whichever order we wish**: indeed, we have $((a^m)^n)^o = (a^{mn})^o$ but also, treating treating a^m as number, $((a^m)^n)^o = (a^m)^{no}$.

The "trick" we did above—treating a^m as number—can also be done when the exponent is a negative integer, rational or real number. And so, we will always have this associativity property. △

The zero exponent. It would be desirable to maintain these three properties when (at least) one of the exponents is zero. In particular, we would like to have $a^m a^0 = a^m$—which leads us to define $a^0 \stackrel{\text{def}}{=} 1$. With this convention, it is obvious that $a^m a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$ still hold when at least one of m, n is zero (and the other is either zero or positive). $(ab)^m = a^m b^m$ also holds when setting $m = 0$.

Negative integers exponents. The next natural step is to define exponentiation to negative integers, in a way that is coherent with these properties. In particular, one would like to have $a^1 a^{-1} = a^0 = 1$, which leads us to define a^{-1} as the multiplicative inverse of a, i.e., $a^{-1} \stackrel{\text{def}}{=} 1/a$. More generally, we require that $a^m a^{-m} = a^0 = 1$, whence we define a^{-m} as the multiplicative inverse of a^m: $a^{-m} \stackrel{\text{def}}{=} 1/a^m$. Note that this means $a^{-m} = (a^m)^{-1}$, because by definition $(a^m)^{-1} = 1/a^m$. Additionally, one can observe that $(a^{-1})^m$ is also the inverse of a^m:

$$a^m(a^{-1})^m = a^m(1/a)^m = \underbrace{aa\cdots a}_{m \text{ times}}\underbrace{(1/a)(1/a)\cdots(1/a)}_{m \text{ times}} = 1 \quad (2.4)$$

As in $\mathbb{R}$ multiplicative inverses are unique, this means we must have $a^{-m} = (a^m)^{-1} = (a^{-1})^m$. Note that this holds even when $m = 0$. This constitutes a very strong hint that we should attempt to generalise the property $(a^m)^n = a^{mn}$ to negative exponents. Another cue in that direction comes from the following observation: the inverse of $1/a$ is a, which means we should have $(a^{-1})^{-1} = a$—but lo and behold, this is exactly what we obtain from multiplying the exponents!

So, to prove that the exponent multiplication rule—$(a^m)^n = a^{mn}$—holds even when at least one of the exponents is negative, let m, n be non-negative integers, and argue by cases:

- $(a^{-m})^n = [(a^{-1})^m]^n = (a^{-1})^{mn} = a^{-mn} = a^{(-m)n}$
- $(a^m)^{-n} = [(a^m)^n]^{-1} = (a^{mn})^{-1} = a^{-mn} = a^{m(-n)}$
- $(a^{-m})^{-n} = \{[(a^m)^{-1}]^{-1}\}^n = (a^m)^n = a^{(-m)(-n)}$

Next comes the exponent sum rule, $a^m a^n = a^{m+n}$—which also holds for negative exponents. We prove so by again reasoning by cases (let again m, n be non-negative integers):

- $a^{-m}a^n = (a^{-1})^m a^n$. As $a^{-1}a = 1$, we have:

$$\begin{cases} (a^{-1})^m a^n = a^0 = a^{-m+n} & \text{if } m = n \\ (a^{-1})^m a^n = (a^{-1})^{m-n} = a^{-m+n} & \text{if } m > n \\ (a^{-1})^m a^n = a^{n-m} = a^{-m+n} & \text{if } m < n \end{cases} \quad (2.5)$$

  But in either case, the property holds.
- $a^m a^{-n} = a^m(a^{-1})^n$. Reasoning in a similar manner as above, we have:

$$\begin{cases} a^m(a^{-1})^n = a^0 = a^{m+(-n)} & \text{if } m = n \\ a^m(a^{-1})^n = (a^{-1})^{n-m} = a^{m+(-n)} & \text{if } m < n \\ a^m(a^{-1})^n = a^{m-n} = a^{m+(-n)} & \text{if } m > n \end{cases} \quad (2.6)$$

But again in either case, the property holds.

- $a^{-m}a^{-n} = (a^{-1})^m(a^{-1})^n = (a^{-1})^{m+n} = a^{-m+(-n)}$

Lastly, to show that the basis multiplication rule, $(ab)^m = a^m b^m$, also holds for negative exponents, requires observing that $(ab)^{-1} = b^{-1}a^{-1}$, because $abb^{-1}a^{-1} = a1a^{-1} = 1$. Now we have:

$$(ab)^{-m} = [(ab)^{-1}]^m = [b^{-1}a^{-1}]^m = (b^{-1})^m(a^{-1})^m = a^{-m}b^{-m} \tag{2.7}$$

Division. The above properties allow us to relate easily division with integer exponentiation. Let a, b, m, n be integers, with a, b nonzero. We have:

$$\begin{aligned}\left(\frac{a}{b}\right)^n = \left(a \times \frac{1}{b}\right)^n = a^n\left(\frac{1}{b}\right)^n = a^n(b^{-1})^n \\ = a^n b^{-n} = a^n\frac{1}{b^n} = \frac{a^n}{b^n}\end{aligned} \tag{2.8}$$

And also:

$$\frac{a^m}{a^n} = a^m \times \frac{1}{a^n} = a^m a^{-n} = a^{m-n} \tag{2.9}$$

3 Rational Exponents

Let a be a nonzero real, and m/n be (a fraction representing) a rational number. When thinking about how to define $a^{m/n}$, one arrives quickly at the desirability of one such definition verifying the following properties:

1. If m/n is actually an integer, let us say k, then we must have $a^{m/n} = a^k$. After all, we want the operation of exponentiation to a rational exponent to *extend* the operation of exponentiation to a integer exponent.
2. $a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}$. If this did not hold, the operation of exponentiation to a rational exponent clearly could not satisfy a property equivalent to (2.2) for the operation of exponentiation to a integer exponent.
3. If m/n and o/p are equivalent fractions (i.e., they represent the same rational number), then $a^{m/n} = a^{o/p}$. Without this, the operation of exponentiation to a rational exponent would not be well-defined.

Note that setting $m = n$ in property 2, we obtain: $a^{n/n} = (a^{1/n})^n = (a^n)^{1/n}$. Assuming a stronger version of it—namely, adding the requirement that $a^{n/n} = a$ (which is a particular case of property 1, for $k = 1$)—we can prove property 1 for any general integer k:

Lemma 3.1 (Property 1). *If m/n is an integer, say k, then $a^{m/n} = a^k$.*

Proof. We have $m/n = k \Leftrightarrow m = nk$. Thus $a^{m/n} = a^{(nk)/n} = (a^{1/n})^{nk}$, and by the properties of integer exponents, we can write this as $\{[(a^{1/n})]^n\}^k = a^k$. ■

With the same assumption, we can also prove property 3:

Lemma 3.2 (Property 3). *If m/n and o/p are two equivalent fractions, with n, p positive, then $a^{m/n} = a^{o/p}$.*

Proof. If m/n and o/p are equivalent, this means that $mp = no$. We have: $a^{m/n} = [(a^{m/n})^p]^{1/p} = (a^{(pm)/n})^{1/p} = (a^{(no)/n})^{1/p} = (a^o)^{1/p} = a^{o/p}$, where the first equality follows from the stronger form of property 2, the second is due to lemma 3.3 below, the third comes from the condition of fraction equivalence, the fourth follows from lemma 3.1, and the fifth from the original form of property 2.[3] ■

Lemma 3.3. *Given integers m, n, l, with n positive, we have $(a^{m/n})^l = (a^l)^{m/n} = a^{(lm)/n}$.*

Proof. $(a^{m/n})^l = [(a^{1/n})^m]^l$, which by the properties of exponentiation to an integer exponent, is equal to $(a^{1/n})^{lm}$, which equals $a^{(lm)/n}$—where both equalities follow from (the original form of) property 2. And similarly, $(a^l)^{m/n} = [(a^l)^m]^{1/n} = (a^{lm})^{1/n} = a^{(lm)/n}$. ■

This means that after having a tentative definition of exponentiation to a rational exponent, we only need to check for the strengthened version of property 2:

Proposition 3.4. *If the definition of exponentiation to a rational exponent verifies $a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}$, and for $m = n$, $a^{m/n} = a$, then it verifies properties 1 and 3 above (in addition to trivially verifying property 2).*

Proof. Immediate from the previous lemmas. ■

Before moving on to define exponentiation with a rational exponent, note an easy corollary to lemma 3.3: as n must be positive, when $l = -1$, we have

Corollary 3.5. *Given integers m, n, with n positive, we have $(a^{m/n})^{-1} = (a^{-1})^{m/n} = a^{-m/n} = a^{(-m)/n}$.*

Let us now tackle the question of *actually defining* exponentiation to a rational exponent. As noted above, setting $m = n$ in condition 1 shows we want to have $(a^{1/n})^n = a$, which means that we must set $a^{1/n} \stackrel{\text{def}}{=} \sqrt[n]{a}$—which not possible if $a < 0$ and n is even. This violates property 2, because reversing the order of the exponents, we obtain

$(a^n)^{1/n}$ which is defined *even when* $a < 0$, because as n is even, a^n will be positive—and hence $\sqrt[n]{a^n}$ will be defined (and indeed, it will be equal to $|a|$). And this is not the only thing that can go wrong: consider, for instance, the case of $(-8)^{1/3}$. On the one hand, this equals $\sqrt[3]{-8} = -2$. On the other, consider $(-8)^{2/6}$: it can either be equal to 2 (rewriting it as $[(-8)^2]^{1/6} = \sqrt[6]{(-8)^2} = \sqrt[6]{64} = 2$), or it can be undefined in the reals (rewriting it as $[(-8)^{1/6}]^2 = (\sqrt[6]{-8})^2$)—thus violating property 2. Which of course means we have violated property 3, for $1/3$ and $2/6$ are equivalent fractions.

Hence, even though exponentiation of a negative real number to a rational exponent may, in certain circumstances make sense, we establish that:

In the general case, exponentiation of a real number to a rational power is defined only for POSITIVE real numbers.

We now define exponentiation with a rational power as follows: given integers m, n, with n positive, we have:

$$a^{m/n} \stackrel{\text{def}}{=} \sqrt[n]{a^m} \qquad (a > 0) \tag{3.6}$$

This means, in particular, that $a^{1/n} = \sqrt[n]{a}$. n must be positive as it is the index of a root—but this causes no trouble, as any rational number can be expressed via a fraction with a positive denominator, and property 3—which we will verify momentarily—ensures that the end result is the same, regardless of the fraction used. Now, by proposition 3.4, besides property 2, we need to check that $a^{n/n} = a$. This translates to $\sqrt[n]{a^n}$, which, given that $a > 0$, is always equal to a. To show that property 2 holds, we need to show that we always have $(a^{1/n})^m = (a^m)^{1/n}$, i.e., $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$. This is taken care of by lemma 3.10, which requires a couple of previous results.

Lemma 3.7. *Let n be a positive integer, and let a, b be reals such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are defined. Then $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ always holds.*

Proof. From the properties of integer exponentiation shown on the previous section, we have: $(\sqrt[n]{a}\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$, and also $(\sqrt[n]{ab})^n = ab$. From both conditions we could have $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$, but also, if n is even, $\sqrt[n]{a}\sqrt[n]{b} = -\sqrt[n]{ab}$. But the latter condition is impossible, because $\sqrt[n]{ab}$ and $\sqrt[n]{a}\sqrt[n]{b}$ always have the same sign, *regardless of whether n is odd or even*: $\sqrt[n]{ab}$ is negative if n is odd, and a and b have different signs (one positive and other negative). For $\sqrt[n]{a}\sqrt[n]{b}$ to be negative, one of the roots has to be positive, and the other negative. This is only possible if n is odd, and one of a, b is negative, and the other positive—and so $\sqrt[n]{a}\sqrt[n]{b}$ is negative if and only if $\sqrt[n]{ab}$ is negative. Obviously, $\sqrt[n]{a}\sqrt[n]{b} = 0$ if and only if $\sqrt[n]{ab} = 0$. Hence, $\sqrt[n]{a}\sqrt[n]{b}$ is positive if and only if $\sqrt[n]{ab}$ is positive—and the conclusion that $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ is now immediate. ■

Corollary 3.8. $\sqrt[n]{a/b} = \sqrt[n]{a}/\sqrt[n]{b}$.

Proof.

$$\sqrt[n]{\frac{a}{b}} = \sqrt[n]{a \times \frac{1}{b}} = \sqrt[n]{a}\sqrt[n]{\frac{1}{b}} \tag{3.9}$$

We have $(\sqrt[n]{1/b})^n = 1/b = (1/\sqrt[n]{b})^n$, from whence one concludes that $\sqrt[n]{1/b} = 1/\sqrt[n]{b}$. This together with (3.9) yields the desired conclusion:

$$\sqrt[n]{a}\sqrt[n]{\frac{1}{b}} = \sqrt[n]{a}\frac{1}{\sqrt[n]{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

■

Lemma 3.10 (Property 2). *For any integer* m*, we have* $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$.

Proof. If $m = 0$ the result is obvious. If m is positive, then the case $m = 1$ is trivial, and the case $m = 2$ follows from lemma 3.7 setting $b = a$. The case for an arbitrary $m > 2$ now follows via a simple inductive argument: if for an arbitrary m we have $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$, then for $m+1$ comes: $(\sqrt[n]{a})^{m+1} = (\sqrt[n]{a})^m(\sqrt[n]{a}) = \sqrt[n]{a^m}\sqrt[n]{a} = \sqrt[n]{a^{m+1}}$.

If m is negative, then $(\sqrt[n]{a})^m = (1/\sqrt[n]{a})^{-m}$, which by the previous corollary is equal to $(\sqrt[n]{1/a})^{-m}$, i.e., $(\sqrt[n]{a^{-1}})^{-m}$. As $-m$ is positive, by the result of the above paragraph, this is the same as $(\sqrt[n]{(a^{-1})^{-m}}) = \sqrt[n]{a^m}$. ■

So definition 3.6 satisfies the stronger version of property 2, which per proposition 3.4, entails that it verifies properties 1 and 3 as well. And now that we have shown that definition 3.6 satisfies the three required conditions, we can proceed to prove the same three properties of exponentiation with integer exponents—exponent addition, exponent multiplication, and basis multiplication—to the case of rational exponents.

First, we have the rule for adding exponents. As we can always assume the denominators are equal, we have:

$$x^{a/c}x^{b/c} = (x^{1/c})^a(x^{1/c})^b = (x^{1/c})^{a+b} = x^{(a+b)/c} \tag{3.11}$$

where in the middle step we have used the property of exponent addition for integer exponentiation.

Next comes the rule for multiplying exponents—but a previous result is required.

Lemma 3.12. *Let* x *be a real number, and* a, b *two positive integers. We have* $\sqrt[a]{\sqrt[b]{x}} = \sqrt[b]{\sqrt[a]{x}} = \sqrt[ab]{x}$.

Proof. Let $y = \sqrt[a]{\sqrt[b]{x}}$. As x is a positive real, y will also be a positive real. And so $y = \sqrt[a]{\sqrt[b]{x}} \Leftrightarrow y^a = \sqrt[b]{x} \Leftrightarrow (y^a)^b = x$. But then $x = (y^b)^a$ holds, from which we obtain $y = \sqrt[b]{\sqrt[a]{x}}$. And finally, via (2.2) we also have $x = y^{ab} \Leftrightarrow y = \sqrt[ab]{x}$. ■

Restated in the language of rational exponents, this result shows that we have: $(x^{1/a})^{1/b} = (x^{1/b})^{1/a} = x^{1/(ab)}$ (although keep in mind that we established above that rational exponentiation is defined only for positive basis).

We can now derive the exponent multiplication rule:

$$\begin{aligned}(x^{m/n})^{o/p} &= \{[(x^m)^{1/n}]^{1/p}\}^o \\ &= [(x^m)^{1/(np)}]^o = (x^{m/(np)})^o = x^{(mo)/(np)}\end{aligned} \tag{3.13}$$

The first equality uses property 2, the second lemma 3.12 (and exponent associativity, cf. remark 2.3), and the third and forth, lemma 3.3.

Finally, the basis multiplication rule:

$$\begin{aligned}(ab)^{m/n} &= [(ab)^{1/n}]^m = [a^{1/n}b^{1/n}]^m \\ &= (a^{1/n})^m(b^{1/n})^m = a^{m/n}b^{m/n}\end{aligned} \tag{3.14}$$

where the first and forth equalities use property 2, the second uses lemma 3.7, and the third the exponentiation of product rule for integer exponents.

Division. Just as in §2, we can derive similar properties for division with rational exponents, as those derived for integer exponents, reasoning similarly. That is to say, we get the following analogues of (2.8) and (2.9) (where μ, ν are rational numbers):

$$\begin{aligned}\left(\frac{a}{b}\right)^\nu &= \left(a \times \frac{1}{b}\right)^\nu = a^\nu \left(\frac{1}{b}\right)^\nu \\ &= a^\nu (b^{-1})^\nu = a^\nu b^{-\nu} = a^\nu \times \frac{1}{b^\nu} = \frac{a^\nu}{b^\nu}\end{aligned} \tag{3.15}$$

And also:

$$\frac{a^\mu}{a^\nu} = a^\mu \times \frac{1}{a^\nu} = a^\mu a^{-\nu} = a^{\mu-\nu} \tag{3.16}$$

4 Real Exponents

To show the same three properties when not just the base, but also the exponents are real numbers, we need to first define a^x, when x is an arbitrary real number (as usual, the base a is a nonzero real number). And to do so, requires understanding one of the ways in which the reals numbers can be constructed, taking the rationals as a starting point. The details are well beyond the scope of the present text, but the idea is to represent numbers that cannot be expressed as a ratio of integer quantities—and hence, are not rational numbers—as *sequences* of rational numbers, that "converge," in a specific sense, to the real number we want to represent.[4] These sequences are called **Cauchy sequences**, and the important points for our current endeavour are:

- For any real x, there exists at least one sequence of *rational* numbers, $\{x_n\}$ that converges to x (this is denoted $x_n \to x$).
- Given two real numbers *defined by their respective Cauchy sequences*, we add and multiply them by adding and multiplying the respective sequences, term-wise.

We now define exponentiation to a real power as follows (where x and $\{x_n\}$ are as above):

$$a^x \stackrel{\text{def}}{=} \lim_{n \to +\infty} a^{x_n} \tag{4.1}$$

To get a feeling of why this definition makes sense, let m/n be a rational number, i.e., m, n are integers, and $n \neq 0$. Then one sequence of rationals that converges to m/n is the constant sequence, namely the sequence that has all terms $x_1, x_2, \ldots$ equal to m/n. Then the above limit converges to $a^{m/n}$. But as in the previous sections, we must show that this definition is proper. Which means two things: first, we must show that when $\{x_n\}$ is a Cauchy sequence (i.e., a real number), then so is $\{a^{x_n}\}$; and second, given that for any real number x, one can always find more than one Cauchy sequence that converges to x, we need to show that the limit above remains the same *for any* such Cauchy sequence. This will take *quite* a bit of mathematical labour.

To begin with, we require the following lemma, which we will state without proof.[5]

Lemma 4.2. *Let* a, b, k *be real numbers, with* $k \neq 0$. *Then if* $a < b$*, we have:*

- $ka < kb$*, if* $k > 0$.
- $ka > kb$*, if* $k < 0$.

We require some other ancillary results.

Lemma 4.3. *Let* a, b, c, d *be real numbers, with* b, c *positive. Then if* $a < b$ *and* $c < d$ *hold, so does* $ac < bd$.

Proof. $a < b \Leftrightarrow ac < bc$, and $c < d \Leftrightarrow bc < bd$, both by lemma 4.2. The result now follows from the transitivity of $<$. ∎

Lemma 4.4. *Let x be a positive real. If $x > 1$, then for any positive integer p, we have $x^p > x^{p-1} > \cdots > x > 1$. If $x < 1$ is the case, then $0 < x^p < x^{p-1} < \cdots < x < 1$.*

Proof. By induction. For the first case, for $p = 1$ it is trivially true that $x^1 > 1$. Assuming that for an arbitrary p we have $x^p > x^{p-1} > \cdots > x > 1$, for $p+1$ we obtain, via lemma 4.2: $x^p > x^{p-1} \Leftrightarrow x^{p+1} > x^p$—and by the transitivity of $<$, we conclude that $x^{p+1} > x^p > x^{p-1} > \cdots > x > 1$ also holds. The proof for the case $x < 1$ is similar, with an extra remark: we can never have $x^p = 0$, for as $\mathbb{R}$ has no zero divisors, that would entail that $x = 0$, which goes against our hypothesis that x is a positive real. ∎

Lemma 4.5. *For any positive real x, we have $x^{1/n} \to 1$.*

The proof of this lemma is rather long, and is thus relegated to §A.

Lemma 4.6. *Let $n \geq 1$ an integer, and x a non-negative real. Then $0 \leq x \leq 1$ if and only if $0 \leq x^{1/n} \leq 1$. In particular, $0 < x < 1$ if and only if $0 < x^{1/n} < 1$.*

Proof. Begin by observing that if $x = 0$ or $x = 1$, the result is obvious, because:

- $x = 0$ if and only if $x^{1/n} = 0$, because the latter condition is equivalent to $x = 0^n$, which means $x = 0$.
- $x = 1$ if and only if $x^{1/n} = 1$, because the latter condition is equivalent to $x = 1^n$, which means $x = 1$.

Thus, we need only show that $0 < x < 1$ if and only if $0 < x^{1/n} < 1$. Let $y = x^{1/n} \Leftrightarrow y^n = x$. ($\rightarrow$) To derive a contradiction, assume that $0 < x < 1$ but that $y > 1$. By lemma 4.4, we also have $y^n > 1$, i.e., $x > 1$, which is a contradiction. ($\leftarrow$) We assume $0 < y < 1$. Again by lemma 4.4, we also have $0 < y^n < 1$, i.e., $0 < x < 1$. ∎

Corollary 4.7. *Let $n \geq 1$ an integer, and x a non-negative real. Then $x > 1$ if and only if $x^{1/n} > 1$.*

Proof. Immediate from the negation of both sides of the equivalence (biconditional) of lemma 4.6. ∎

Lemma 4.8. *Let x be a positive real, and let q, r be integers. If $x > 1$, then $x^q > x^r$ if and only if $q > r$. If $0 < x < 1$, then $x^q > x^r$ if and only if $q < r$.*

Proof. Let $x > 1$. ($\rightarrow$) We assume that $x^q > x^r$—from which follow several cases:

- q and r are both positive. By lemma 4.4 we have:

$$1 < x < \cdots < x^{r-1} < x^r < x^{r+1} < \cdots < x^q$$

  from whence follows that $q > r$.

Before continuing, note the following: we have $q = r$ if and only if $x^q = x^r$: the forward direction follows from the fact that exponentiation to a integer power is well-defined, cf. §2. The backward direction follows via the contrapositive: if we had $q \neq r$, then by a similar reasoning to what we did above, we would conclude that either $x^q > x^r$ or $x^q < x^r$—which is a contradiction either way. Hence, as we assume that $x^q > x^r$, we cannot have $q = r$—which means that whenever $q > r$ is not the case, then we must have $q < r$ (and vice-versa).

- $r = 0$. We have $x^q > x^r = x^0 = 1$, and if it were the case that $q < r$, i.e., that q was negative, then $-q$ would be positive, and by lemma 4.3 we would have, as $x > 1$, that $x^{-q} > 1 \Leftrightarrow 1/x^{-q} < 1/1 \Leftrightarrow x^q < 1$, which is contradictory. Thus we conclude $q > r$.
- $q = 0$. We have $1 = x^q > x^r$, and if it were the case that $q < r$, i.e., that r was positive, then by lemma 4.3 we would have, as $x > 1$, that $x^r > 1$, which is contradictory. Thus we conclude $q > r$.
- q and r are both negative. Then we can write $q = -q'$ and $r = -r'$, with q', r' positive integers. We have $x^q > x^r \Leftrightarrow x^{-q'} > x^{-r'} \Leftrightarrow 1/x^{q'} > 1/x^{r'} \Leftrightarrow x^{q'} < x^{r'}$, from which, by what we proved in the first bullet above, follows that $q' < r' \Leftrightarrow -q' > -r' \Leftrightarrow q > r$.

($\Leftarrow$) We assume that $q > r$—and again have several cases:

- q and r are both positive, or q is positive and $r = 0$. Then as $x > 1$, $x^q > x^r$ follows directly from lemma 4.4.
- $q = 0$ and r negative. As r is negative, we have $x^r = (1/x)^{-r}$, with $-r$ positive and $1/x < 1$. Thus by lemma 4.4 comes that $1 > (1/x)^{-r} = x^r$, from whence follows $x^q = x^0 = 1 > x^r$.
- q and r are both negative. Then we can write $q = -q'$ and $r = -r'$, with q', r' positive integers. We have $q > r \Leftrightarrow -q' > -r' \Leftrightarrow q' < r'$. By the result of the first bullet in this new enumeration, we have $x^{q'} < x^{r'} \Leftrightarrow 1/x^{q'} > 1/x^{r'} \Leftrightarrow x^{-q'} > x^{-r'} \Leftrightarrow x^q > x^r$.

Thus we are done with the $x > 1$ case—so let us now assume that $x < 1$. We have

$$x^q > x^r \Leftrightarrow [(x^{-1})^{-1}]^q > [(x^{-1})^{-1}]^r$$
$$\Leftrightarrow \left(\frac{1}{x}\right)^{-q} > \left(\frac{1}{x}\right)^{-r}$$

Now $1/x > 1$, so by the first part of this proof, we immediately have $-q > -r \Leftrightarrow q < r$. ■

This is also valid for *rational* exponents:

Proposition 4.9. *Let x be a positive real, and let q, r be rationals. If $x > 1$, then $x^q > x^r$ if and only if $q > r$. If $0 < x < 1$, then $x^q > x^r$ if and only if $q < r$.*

Proof. Write q and r as fractions with a common positive denominator: $q = q'/d$ and $r = r'/d$. Thus, $x^q > x^r \Leftrightarrow (x^{1/d})^{q'} > (x^{1/d})^{r'}$. We have $x > 1$ if and only if $x^{1/d} > 1$ (corollary 4.7) if and only if $q' > r'$ (lemma 4.8), which of course holds if and only if $q > r$.

The reasoning is similar for $x < 1$, but we need lemma 4.6, rather than corollary 4.7. ■

Corollary 4.10. *If $x > 1$ is a real and q is a rational number, then $x^q > 1$ if and only if $q > 0$, and $0 < x^q < 1$ if and only if $q < 0$.*

Corollary 4.11. *If $0 < x < 1$ be is a real and q is a rational number, then $x^q > 1$ if and only if $q < 0$, and $0 < x^q < 1$ if and only if $q > 0$.*

In both corollaries, the first bi-conditional follows from setting $r = 0$ in proposition 4.9. For the second bi-conditional, set $q = 0$ in the same proposition; the result now follows from the fact that $x^r > 0$ always holds (lemma 4.4).

We can now show that definition 4.1 is proper. We split the proof into the next two lemmas.

Lemma 4.12. *Let $x > 0$ be a real, and $\{q_n\}$ a Cauchy sequence of rational terms. Then the sequence $\{x^{q_n}\}$ is also a Cauchy sequence.*

Proof. If $x = 1$ the statement is trivial, and so we are left with two cases: $x > 1$ and $x < 1$. We start with $x > 1$. We have:

$$\left|x^{q_n} - x^{q_m}\right| = \left|x^{q_m}(x^{q_n - q_m} - 1)\right| = \left|x^{q_m}\right| \cdot \left|x^{q_n - q_m} - 1\right|$$

As $\{q_n\}$ is a Cauchy sequence, it is bounded and in particular, it has an upper bound, let us say M. As $x > 1$, by proposition 4.9 we have $x^{q_m} \leq x^M$, and furthermore, as by corollary 4.10, both sides are positive, it is also the case that $\left|x^{q_m}\right| \leq x^M$. Moving on to the next parcel, via lemma 4.5 ($\lim_{k\to\infty} x^{1/k} = 1$), follows that for any real $\varepsilon > 0$, there exists p such that $\left|x^{1/k} - 1\right| < \varepsilon x^{-M}$, for $k \geq p$ (note that this means $1/k \leq 1/p$). And lastly, again due to $\{q_n\}$ being a Cauchy sequence, there exists r such that $|q_n - q_m| < 1/p$, for all $m, n \geq r$. From $|q_n - q_m| < 1/p$ follows that $q_n - q_m < 1/p$. Again by proposition 4.9 we have

$x^{q_n - q_m} < x^{1/p}$. Now we have two possibilities: either $q_n > q_m$ or $q_n < q_m$ (the case $q_n = q_m$ is trivial). If $q_n > q_m$, then corollary 4.10 shows that both members are greater than 1 (and thus positive), and so we have $|x^{q_n-q_m} - 1| < \left|x^{1/p} - 1\right|$. All of which taken together shows that given any $\varepsilon > 0$, there exists r such that, for $m, n \geq r$ we have:

$$|x^{q_n} - x^{q_m}| = |x^{q_m}| \cdot \left|x^{q_n-q_m} - 1\right| < x^M \varepsilon x^{-M} = \varepsilon$$

If $q_n < q_m$, we have:

$$|x^{q_n} - x^{q_m}| = |x^{q_m} - x^{q_n}| = \left|x^{q_n}(x^{q_m-q_n} - 1)\right| = |x^{q_n}| \cdot \left|x^{q_m-q_n} - 1\right|$$

And by reasoning as above, we conclude that for $m, n \geq r$ (all as above), we again have $|x^{q_n-q_m} - 1| < \left|x^{1/p} - 1\right|$. Thus we have shown that for any $\varepsilon > 0$, there is a point from which we have $|x^{q_n} - x^{q_m}| < \varepsilon$—and this establishes that $\{q_n\}$ is indeed a Cauchy sequence.

Now for the $x < 1$ case, we still have $|x^{q_n} - x^{q_m}| = |x^{q_m}| \cdot |x^{q_n-q_m} - 1|$, which we can rewrite as:

$$\left|(1/x)^{-q_m}\right| \cdot \left|(1/x)^{q_m-q_n} - 1\right|$$

As $\{q_n\}$ is bounded, it has a *lower* bound, which will be an upper bound of $\{-q_n\}$—let M be that upper bound. Now $0 < x < 1$ means that $1/x > 1$—and hence, by proposition 4.9 we have $(1/x)^{-q_m} \leq (1/x)^M$, and moreover, by corollary 4.10, both sides are positive, meaning we have $|(1/x)^{-q_m}| \leq (1/x)^M$. Moving on to the next parcel, via lemma 4.5 we have $\lim_{k\to\infty}(1/x)^{1/k} = 1$—and hence, given any $\varepsilon > 0$, there exists p such that $\left|(1/x)^{1/k} - 1\right| < \varepsilon(1/x)^{-M}$, for $k \geq p$. And due to $\{q_n\}$ being a Cauchy sequence, there exists r such that $|q_m - q_n| < 1/p$, for all $m, n \geq r$, which entails that $q_m - q_n < 1/p$. By proposition 4.9 we have $(1/x)^{q_m-q_n} < (1/x)^{1/p}$. Assuming we have $q_m > q_n$, corollary 4.10 shows that both members are greater than 1 (and thus positive), and so we have $|(1/x)^{q_m-q_n} - 1| < \left|(1/x)^{1/p} - 1\right|$ (if $q_m < q_n$, we go around that difficulty the same way we did for the case of $x > 1$, cf. above). All of which taken together shows that given any $\varepsilon > 0$, there exists r such that, for $m, n \geq r$ we have:

$$|x^{q_n} - x^{q_m}| = |x^{q_m}| \cdot \left|x^{q_n-q_m} - 1\right| < (1/x)^M \varepsilon (1/x)^{-M} = \varepsilon$$

which shows that $\{x^{q_n}\}$ is a Cauchy sequence also when $x > 1$. And we are done. ■

Lemma 4.13. *Let $x > 0$ be a real, and $\{q_n\}$ and $\{q'_n\}$ two Cauchy sequences (of rational terms) that represent x. Then* $\lim_{n\to\infty} x^{q_n} = \lim_{n\to\infty} x^{q'_n}$.

Proof. By the properties of the limits of sequences, we have $\lim_{n\to\infty} x^{q_n} = \lim_{n\to\infty} x^{q_n - q'_n} \cdot x^{q'_n} = \lim_{n\to\infty} x^{q_n - q'_n} \cdot \lim_{n\to\infty} x^{q'_n}$, because by the previous lemma, both latter limits exist. Which means we need only show that $\lim_{n\to\infty} x^{q_n - q'_n} = 1$. This is obvious when $x = 1$, which means we are again left with two cases, viz. $x > 1$ and $x < 1$. We take $x > 1$ first.

From lemma 4.5 ($\lim_{k\to\infty} x^{1/k} = 1$), follows that for any real $\varepsilon > 0$, there exists r such that $\left|x^{1/k} - 1\right| < \varepsilon$, for $k \geq r$. This latter condition entails, in particular, that $x^{1/k} < 1+\varepsilon$. And from the usual properties of limits of sequences, we can also derive from lemma 4.5 that $\lim_{k\to\infty} x^{-1/k} = 1$—which, via a similar reasoning, means that there exists s such that $1 - \varepsilon < x^{-1/k}$, for $k \geq s$. Let $t = \max\{r, s\}$. As $\{q_n\}$ and $\{q'_n\}$ are Cauchy sequences with the same limit, we have $\lim_{n\to\infty} q_n - q'_n = 0$. This means that there exists p such that $-1/t < q_n - q'_n < 1/t$, for $n \geq p$. By proposition 4.9, this is equivalent to $x^{-1/t} < x^{q_n - q'_n} < x^{1/t}$. This is to say that, for $n \geq p$, we have:

$$1 - \varepsilon < x^{-1/t} < x^{q_n - q'_n} < x^{1/t} < 1 + \varepsilon$$

As ε is positive but arbitrary, this shows that $\lim_{n\to\infty} x^{q_n - q'_n} = 1$.

The case for $x < 1$ is similar, but for a few changes: we want r such that $1 - \varepsilon < x^{1/k}$ for $k \geq r$, and s such that $x^{-1/k} < 1 + \varepsilon$ for $k \geq s$. Then from $-1/t < q_n - q'_n < 1/t$ (for $n \geq p$) by proposition 4.9 comes $x^{-1/t} > x^{q_n - q'_n} > x^{1/t}$, and from this the conclusion (*idem*):

$$1 + \varepsilon > x^{-1/t} > x^{q_n - q'_n} > x^{1/t} > 1 - \varepsilon$$

Again showing that $\lim_{n\to\infty} x^{q_n - q'_n} = 1$, now for $x < 1$. ■

Lemmas 4.12 and 4.13 together establish that definition 4.1 is proper. Thus, we can now proceed to prove the usual three properties of exponentiation—viz., exponent addition and multiplication, and basis multiplication—also for the case of real exponents.

Let x, y, q, r be real numbers, with x, y positive, and let $\{q_n\} \to q$ and $\{r_n\} \to r$ be Cauchy sequences representing q and r, respectively. First comes the rule to add exponents. We have:

$$x^{q+r} = \lim_{n\to\infty} x^{q_n + r_n} = \lim_{n\to\infty} x^{q_n} x^{r_n} = \lim_{n\to\infty} x^{q_n} \lim_{n\to\infty} x^{r_n} = x^q x^r$$

where the second equality comes from the exponent addition rule for rational exponents, and the third equality from the usual properties of limits of sequences.

Next comes the rule of exponent multiplication. We have

$$(x^q)^r = \lim_{m\to\infty} \left(\lim_{n\to\infty} x^{q_n} \right)^{r_m}$$

In §B it is shown that we have $(\lim_{n\to\infty} x^{q_n})^{r_m} = \lim_{n\to\infty}(x^{q_n})^{r_m}$ (note that m is fixed). And each of the terms on the right hand side is, by the properties of exponentiation to a rational exponent, equal to $x^{q_n r_m}$. Thus, r_m is a constant value, and so as $\{q_n\}$ is a Cauchy sequence, so is $\{q_n r_m\}$—and moreover, we have $\lim_{n\to\infty} q_n r_m = q r_m$. That is, we have:

$$(x^q)^r = \lim_{m\to\infty}\left(\lim_{n\to\infty} x^{q_n}\right)^{r_m} = \lim_{m\to\infty}\left(\lim_{n\to\infty} x^{q_n r_m}\right) = \lim_{m\to\infty} x^{q r_m} = x^{qr}$$

where the fourth equality comes from the fact that, as $\{r_m\} \to r$, then $\{q r_m\} \to qr$ (as $\{r_m\}$ is Cauchy, so is $\{q r_m\}$).

Lastly, comes the basis multiplication rule. We have:

$$(xy)^q = \lim_{n\to\infty}(xy)^{q_n} = \lim_{n\to\infty} x^{q_n} y^{q_n}$$

and because $\lim_{n\to\infty} x^{q_n} = x^q$ and $\lim_{n\to\infty} x^{r_n} = x^r$, it follows that $\lim_{n\to\infty} x^{q_n} y^{q_n} = (x^q)(y^q)$.

A Proof of lemma 4.5

We need some auxiliary results to prove lemma 4.5.

Lemma A.1. *Let $\{x_n\}$ be a bounded sequence and $\{y_n\} \to 0$ another convergent sequence. Then $\{x_n y_n\} \to 0$.*

Proof. Let M be a bound of $\{x_n\}$, which is to say that for all n, we have $|x_n| \leq M$. If $M = 0$, this means $\{x_n\}$ is the all zero sequence, which in turn means the result is obvious. Thus, assume that $M > 0$.[6] Now given any real $\varepsilon > 0$, because $\{y_n\} \to 0$ there exists p such that $|y_n| < \varepsilon M^{-1}$, for $n \geq p$. And thus $|x_n y_n| = |x_n||y_n| < M\varepsilon M^{-1} = \varepsilon$, which establishes that $\{x_n y_n\} \to 0$. ■

Lemma A.2. *Let $\{x_n\}$ be a bounded and monotone sequence. Then it is convergent.*

Proof. Suppose $\{x_n\}$ is an increasing sequence, and let $c = \sup\{x_n\}$. By definition of supremum, for any real $\varepsilon > 0$ there exists p such that $c - \varepsilon < x_p$. And as $\{x_n\}$ is increasing, it is also the case that for all $n \geq p$, we also have $c - \varepsilon < x_n$. But this latter condition implies $c - \varepsilon < x_n < c + \varepsilon \Leftrightarrow |x_n - c| < \varepsilon$, which establishes that $\lim_{n\to\infty} x_n = c$, thus showing that $\{x_n\}$ is convergent.

If $\{x_n\}$ is a decreasing sequence, we set $c = \inf\{x_n\}$ and proceed analogously. ■

Lemma A.3. *Let $0 < x < 1$. Then $\{x^n\} \to 0$.*

Proof. By lemma 4.4 for $x < 1$, $\{x^n\}$ is strictly decreasing, and bounded from below by 0. Thus by lemma A.2 it is convergent—let its limit be l. But then, by the usual limit laws, we have $\lim_{n\to\infty} x^{n+1} = \lim_{n\to\infty} x \cdot x^n = x \lim_{n\to\infty} x^n = xl$. However, $\{x^{n+1}\}$ is just $\{x^n\}$ minus its first term (x_1)—and hence, they must have the same limit. As $x \neq 1$, we have $xl = l \Leftrightarrow l = 0$. ■

Lemma A.4. *The sequence* $\{x^n\}$ *converges to* 0 *if* $|x| < 1$, *converges to* 1 *if* $x = 1$, *and diverges otherwise (i.e., if* $x = -1$ *or* $|x| > 1$*).*

Proof. For $x = 1$, the result is obvious. If $x = -1$, then $\{x_n\}$ has two subsequences with different limits ($\{1\}$ and $\{-1\}$), and is accordingly divergent. If $0 < x < 1$, the result follows from lemma A.3. If $-1 < x < 0$, then $\{x^n\} = \{(-1)^n(-x)^n\} = \{(-1)^n\} \times \{(-x)^n\}$. As $0 < -x < 1$, $\lim_{n\to\infty}(-x)^n = 0$, and as $\{(-1)^n\}$ is bounded, by lemma A.1, $\{(-1)^n(-x)^n\}$ converges to 0.

Finally, if $x > 1$ (resp. $x < -1$), we derive a contradiction by assuming that the limit $\lim_{n\to\infty}\{x^n\}$ exists—let it be equal to l. As $0 < 1/x < 1$ (resp. $-1 < 1/x < 0$), by lemma A.3 (resp. the previous paragraph) $\lim_{n\to\infty}(1/x)^n = 0$. We now have: $\lim_{n\to\infty} x^n(1/x)^n = (\lim_{n\to\infty} x^n)(\lim_{n\to\infty}(1/x)^n) = l \times 0 = 0$. Which is obviously absurd, because all terms of the sequence $\{x^n(1/x)^n\}$ are equal to 1! ■

Lemma A.5. *For any positive reals* M, *and* ε, *there exists a positive integer* n *such that* $M^{1/n} \leq 1 + \varepsilon$.

Proof. Towards a contradiction, write the negation of what the lemma says: there exist $M > 0$ and $\varepsilon > 0$ such that for all positive integers n, $M^{1/n} > 1 + \varepsilon$ holds. This is equivalent to $M > (1+\varepsilon)^n$. And it is impossible, because $\{(1+\varepsilon)^n\}$ is strictly increasing (lemma 4.8), and does not converge (lemma A.4)—and hence by the converse of lemma A.2, it is not bounded. ■

Lemma A.6. *For any positive reals* M, *and* ε, *there exists a positive integer* n *such that* $1 - \varepsilon \leq M^{1/n}$.

Proof. We will again derive a contradiction, and will also start by negating the lemma: there exist $M > 0$ and $\varepsilon > 0$ such that for all positive integers n, $M^{1/n} < 1 - \varepsilon$ holds. This is the same as $M < (1-\varepsilon)^n$, which is impossible:

- if $1 - \varepsilon \in]-1, 1[$, then $\{(1-\varepsilon)^n\} \to 0$ by lemma A.4, and so we can find an n such that $(1-\varepsilon)^n < M/2 < M$, for example.
- if $1 - \varepsilon \in]-\infty, 1]$, then $\{(1-\varepsilon)^n\}$ will alternate between positive and negative values, and thus it is impossible to have $0 < M < (1-\varepsilon)^n$ for all n.

The conclusion now follows. ■

Lemma A.7. *$\{x^{1/n}\}$ is strictly increasing if and only if $0 < x < 1$, and strictly decreasing if and only if $x > 1$.*

Proof. Because both sides of the inequality $x^{1/n} < x^{1/(n+1)}$ are positive, it follows from lemma 4.3 (setting $a = c$ and $b = d$) that we can raise both sides to $n + 1$, yielding $x^{(n+1)/n} < x$. As $(n + 1)/n > 1$, by proposition 4.9 this holds if and only if $0 < x < 1$—establishing that $\{x^{1/n}\}$ is strictly increasing. The case for $x > 1$ is identical. ■

We can finally prove lemma 4.5, which we restate below.

Lemma 4.5. *For any positive real x, we have $x^{1/n} \to 1$.*

Proof. The case $x = 1$ is trivial, so we again have $x > 1$ and $x < 1$—and we start with the former. By lemma A.5 for any real $\varepsilon > 0$ there exists a positive integer N such that $x^{1/N} \leq 1 + \varepsilon$. By lemma A.7 $\{x^n\}$ is strictly decreasing, which means for any $n \geq N$, we also have $x^{1/n} \leq 1 + \varepsilon$. But by corollary 4.7 we also have $x^{1/n} > 1$, for *any* positive integer n. Thus $1 < x^{1/n} \leq 1 + \varepsilon$, which implies that $-\varepsilon < x^{1/n} - 1 \leq \varepsilon$, for $n \geq N$. This shows that $x^{1/n} \to 1$—for $x > 1$.

For $x < 1$, by lemma A.6 for any real $\varepsilon > 0$ there exists a positive integer N such that $1 - \varepsilon \leq x^{1/N}$. By lemma A.7 $\{x^n\}$ is strictly increasing, which means for any $n \geq N$, we also have $1 - \varepsilon \leq x^{1/n}$. But by lemma 4.6, we also have $0 < x^{1/n} < 1$, for *any* positive integer n. Thus $1 - \varepsilon \leq x^{1/n} < 1$, which implies that $-\varepsilon \leq x^{1/n} - 1 < \varepsilon$, for $n \geq N$. This shows that $x^{1/n} \to 1$, now also for $x < 1$. ■

B Sequences and continuous functions

At the end of §4, we postponed to the present section the proof that $(\lim_{n\to\infty} x^{q_n})^{r_m} = \lim_{n\to\infty}(x^{q_n})^{r_m}$. This requires the notion of *continuous functions* from calculus (which we state in the proof below).

Lemma B.1. *Let X be a subset of $\mathbb{R}$ and $f\colon X \to \mathbb{R}$ be a function. The function f is continuous at point $a \in X$ if and only if for every sequence $\{x_n\}$ of points of X that converges to a, we have $f(x_n) \to f(a)$.*

Proof. Let $\{x_n\}$ be a sequence of points of X such that $\{x_n\} \to a$. By definition of convergent sequence, this means that ($\varepsilon \in \mathbb{R}$ and $n, p \in \mathbb{N}$):

$$\forall \varepsilon > 0 \; \exists p \; \forall n \;\; n \geq p \Rightarrow |x_n - a| < \varepsilon \tag{B.2}$$

($\rightarrow$) We assume f is continuous at a, and want to show that this implies $\{f(x_n)\} \rightarrow f(a)$. By definition of continuity, the following holds:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in X\ \ 0 < |x - a| < \delta \Rightarrow |f(x) - a| < \varepsilon \tag{B.3}$$

The condition we want to show—$\{f(x_n)\} \rightarrow f(a)$—translates too:

$$\forall \varepsilon > 0\ \exists p\ \forall n\ \ n \geq p \Rightarrow |f(x_n) - f(a)| < \varepsilon \tag{B.4}$$

The first observation, is that for any n such that $x_n = a$, (B.4) holds trivially, so we can assume $x_n \neq a$. For these, it follows from (B.3) that given any real $\varepsilon > 0$, there exists a real $\delta > 0$ such that $\forall x \in X$, $0 < |x_n - a| < \delta \Rightarrow |f(x_n) - f(a)| < \varepsilon$. And from (B.2) we know that for *that* δ, there exists p such that $n \geq p \Rightarrow |x_n - a| < \delta$. Combining these two statements yields (B.4).

($\leftarrow$) We assume that $\{x_n\} \rightarrow a$ implies $\{f(x_n)\} \rightarrow f(a)$, for any arbitrary sequence $\{x_n\}$ of points of X. We want to show that this in turn implies that f is continuous at a, i.e., that $\lim_{x\rightarrow a} f(x) = f(a)$. We will prove this via the contrapositive, that is, we will assume that f is *not* continuous at a, and show that in such a case, there exists a sequence $\{x_n\}$ such that $\{x_n\} \rightarrow a$ and $\{f(x_n)\} \not\rightarrow f(a)$.

So, if f is not continuous at a, this means that the following condition holds:

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in X\ \ 0 < |x - a| < \delta \wedge |f(x) - a| \geq \varepsilon \tag{B.5}$$

Fix an ε for which (B.5) holds. Then, if we set $\delta = 1$, there exists at least one value of x such that the sub-condition $0 < |x - a| < \delta \wedge |f(x) - a| \geq \varepsilon$ holds—let x_1 equal that value of x. More generally, let x_n be a value of x for which the same sub-condition holds when $\delta = 1/n$. It is clear that $\{x_n\} \rightarrow a$, but $\{f(x_n)\} \not\rightarrow f(a)$, which concludes the proof. ∎

An equivalent formulation of this lemma is that for any sequence $\{x_n\}$ that converges to a point at which a function f is continuous, we have $\lim_{n\rightarrow\infty} f(x_n) = f\left(\lim_{n\rightarrow\infty} x_n\right)$.

We can prove that $(\lim_{n\rightarrow\infty} x^{q_n})^{r_m} = \lim_{n\rightarrow\infty}(x^{q_n})^{r_m}$. Recall that m is fixed, meaning that r_m is constant (cf. the ending of §4, if necessary). Also, x is a positive real. As $r_m \in \mathbb{Q}$, we can write it as a/b, with $a, b \in \mathbb{Z}$ and b positive. Thus, by the definition of exponentiation with a rational exponent (§3), we have $(x^{q_n})^{r_m} = \sqrt[b]{(x^{q_n})^a}$. If we set $f(x) = \sqrt[b]{x^a}$, then $(x^{q_n})^{r_m} = f(x^{q_n})$. Now f is differentiable—and thus continuous—at any $x \in \mathbb{R}^+$, and so:

$$\lim_{n\rightarrow\infty} (x^{q_n})^{r_m} = \lim_{n\rightarrow\infty} f(x^{q_n}) = f\left(\lim_{n\rightarrow\infty} x^{q_n}\right) = \left(\lim_{n\rightarrow\infty} x^{q_n}\right)^{r_m}$$

Notes

1. See https://en.wikipedia.org/wiki/Zero_to_the_power_of_zero (accessed March 23, 2023).
2. As an example, consider that $(-2)^2 = 2^2$, but $-2 \neq 2$.
3. Note that this means that the fifth equality also follows from the *stronger* form of property 2.
4. The commas around the word "converge" are because this sequence of rationals need not converge to a rational number—so, strictly speaking, it can be a *divergent* sequence *in the rationals*. But it always converges to some *real* number.
5. I have a forthcoming manuscript wherein an explicit construction of the reals is presented. In it, I also prove this statement (among several others of similar nature). I will update this paper with the reference once it is ready.
6. M cannot be negative, because the absolute value is always non-negative.