# Sine And Cosine in $\mathbb{R}$:
# From Geometric Definitions To Taylor Formulae

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**Abstract.** This paper bridges the gap between the "geometric definitions" of sine and cosine, based on (tri)angles and/or the trigonometric circle, and what is usually presented in, e.g., undergraduate mathematics as their rigorous definitions, based on Taylor polynomials and/or the exponential function.

**Prerequisites.** The reader is expected to be comfortable with topics usually taught in freshman calculus courses, viz., sequences, (one variable) functions, and taking limits of both, continuity, differentiation and power/Taylor series.

**Keywords:** trigonometry, sine, cosine, differentiation, Taylor series.

## 1 Introduction

Most people learn trigonometry in the context of geometry: angles and triangles, and the trigonometric circle, and define sine and cosine using these concepts. The problem with this approach, is that it is not rigorous. Conversely, rigorous analytical definitions of these functions are often anything but intuitive: typically, one either starts from the fact that for sin and cos, we have $f'' = -f$, and then derives their Taylor expansion, which is then used *as the definition* sin and cos; or one first defines the complex exponential function, and then defines sine and cosine in terms of that exponential.[1] The problem is that it is then well-nigh impossible to see how the sine and cosine functions defined in this manner coincide with their geometrical counterparts. To be sure, one can prove that these analytical definitions have some of the same properties as the geometrical sine and cosine,[2] but this does not prove that the analytically defined functions actually describe the original geometrical "reality."

Such a proof is not easily found on the literature—I could only find a proof sketch from Spivak, cf. the "Credits" paragraph, below—and hence I purport to provide one here. I shall stick with sin and cos only, because all other trigonometric functions can be derived from them.

**Strategy.** We will begin by defining the cosine function, in a manner that agrees with our geometric understanding of that function, but made rigorous by the use of the

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integral. This will be done, at first, for angles on a limited interval, but the definition will then be broadened to any real number. Having defined cosine, defining sine from it is straightforward. These functions will then be proved to be continuous, and indefinitely differentiable. This will be the task of §2.

After that is done, we will leverage $\sin'$ and $\cos'$ to derive the Taylor series for each function. This will show that the formulae that in calculus courses is usually taken to be the *definition* of sine and cosine, do indeed coincide with the “geometric sine” and “geometric cosine” that one usually learns in high-school trigonometry. This is done in §3.

Following that, we will derive, taking the Taylor polynomials as the starting point, both some expected properties—e.g., $\sin^2 x + \cos^2 x = 1$—as well as some well-known formulas, such as the sine and cosine of sums, without any appeal to geometric intuition. This section, §4, will end with a brief and informal discussion about how the Taylor polynomial of sine can be used to rigorously define $\pi$, and how from there one can easily show that sine and cosine are periodic functions.

The paper concludes with some remarks about the formal and informal usage of the number $\pi$, in §5. In the appendix, proofs for some theorems required throughout the text are given.

**Credits.** The idea of defining cosine in terms of the integral is largely drawn from Spivak [3, §15], although we provide some of the details absent therein. The presentation of the Taylor polynomials takes some inspiration from Sarrico [2, §9].

# 2 From Geometry To Formal Definitions

We will start with cosine, because once that is defined, is trivial to define sine from it. The idea is to go from thinking of cosine as a function that applies to an *angle*, to thinking of it as a function that applies to *an arbitrary real number*. So starting with angles, if one measures them counter-clockwise[3] from the positive semi-axis of the abscissae, then one can identity the amplitude of an angle with the corresponding length of the arc of the unit circumference delimited by that angle—which is, of course, how one defines *radians*. In particular, as the whole unit circumference has length $2\pi$, it corresponds to an amplitude of $2\pi$ radians. Also, when the sector is the entire unit circle, its area is $\pi$, i.e., half of the length of the corresponding arc.

Accordingly, given an angle $\theta$ as in figure 1, the corresponding arc has length $\theta$ (in blue), and the corresponding sector (purple-ish shade) has area $\theta/2$. The reader likely learned that $\cos\theta$ and $\sin\theta$ are defined in such a way as to make $(\cos\theta, \sin\theta)$ be the coordinates of point P—i.e., $(x, y) = (\cos\theta, \sin\theta)$. Towards a more formal approach, one could try to express the length of arc PR in terms of $x$. Integral calculus would certainly be up to the task, but one usually learns integration by reasoning over areas, rather than lengths—and thus, we will instead express the *area* of the sector

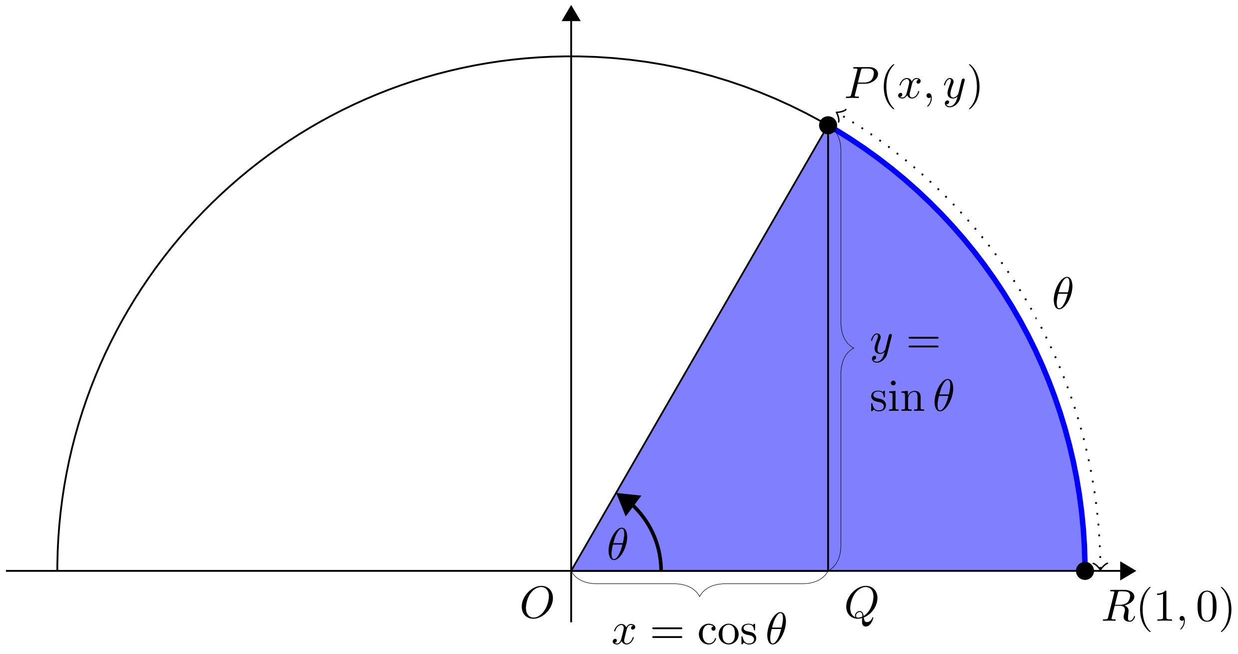


**Figure 1:** Sine and cosine on the trigonometric circle.

corresponding to the arc $PR$ (i.e., $POR$) in terms of $x$.

Let $A$ be the function that, given $x \in [-1, 1]$, returns the area corresponding to the sector defined by arc $PR$ (where $P$, $R$ and $x$ are as above). We define $A\colon [-1, 1] \to [0, \pi/2]$ as follows:

$$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}dt$$

To see that it indeed computes the desired area, observe that the left parcel computes the area of right triangle $PQO$ in figure 1, and the integral computes the remaining area of that sector, that is "above" line segment $QR$. Observe that this formula also holds for negative values of $x$ (i.e., when $\theta > \pi/2$), because then the left parcel is negative, and corresponds exactly to the area that has to be *subtracted* from the integral to obtain the area of the unit circle sector—cf. the triangle $PQO$ in figure 2.

Thus, we want the cosine of an angle $\theta$ to be the value of $x$ such that $A(x) = \theta/2$. After which, defining the sine is trivial. We have the following definition.

**Definition 2.1.** *Let* $\theta \in [0, \pi]$. $\cos\theta$ *is the (unique) value such that* $A(\cos\theta) = \theta/2$. *And* $\sin\theta \overset{\text{def}}{=} \sqrt{1-\cos^2\theta}$.

For this definition to even make sense, we must show that for any $y \in [0, \pi/2]$ there *exists* such a number $x \in [-1, 1]$ verifying $A(x) = y$. For this we must differentiate $A$.

We start by observing that the derivative is only defined for $x \in ]-1, 1[$, because the derivative of $\sqrt{x}$ is not defined at $x = 0$—hence, that of $\sqrt{1-x^2}$ is not defined

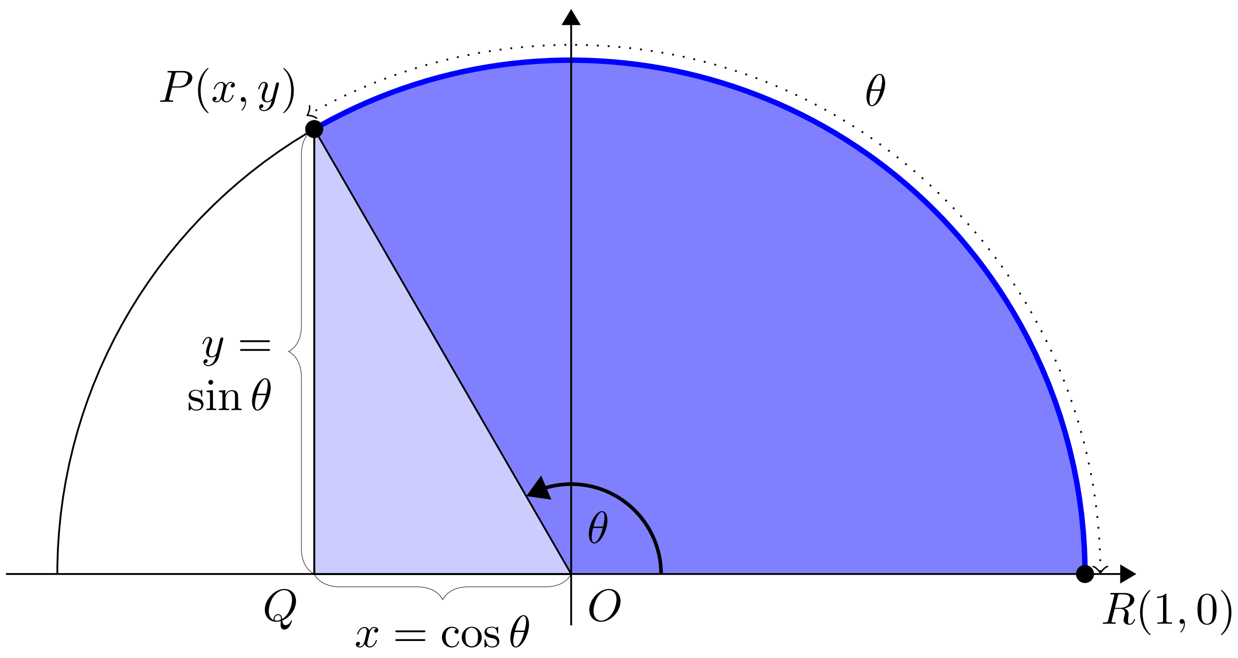


**Figure 2:** When $x$ is negative.

for $x = \pm 1$.[4] Using the usual differentiation rules, together with the Fundamental Theorem of Calculus, we have:

$$
\begin{aligned}
A'(x) &= \frac{1}{2}\left(x \cdot \frac{-2x}{2\sqrt{1-x^2}} + \sqrt{1-x^2}\right) - \sqrt{1-x^2} \\
&= \frac{1}{2}\left(\frac{-x^2 + (1-x^2)}{\sqrt{1-x^2}}\right) - \sqrt{1-x^2} \\
&= \frac{1-2x^2}{2\sqrt{1-x^2}} - \sqrt{1-x^2} \\
&= \frac{1-2x^2-2(1-x^2)}{2\sqrt{1-x^2}} = \frac{-1}{2\sqrt{1-x^2}}
\end{aligned}
$$

From a well-known theorem in calculus, the fact that $A$ is differentiable in $]-1, 1[$ means it is also continuous on that interval[5]—but we can also prove continuity at the extremes. Recall that by the Fundamental Theorem Calculus, we know that the integral in the definition of $A$ is continuous in $[-1, 1]$. We have:

$$
\begin{aligned}
\lim_{x \to -1} A(x) &= \lim_{x \to -1} \frac{x\sqrt{1-x^2}}{2} + \lim_{x \to -1} \int_x^1 \sqrt{1-t^2}dt \\
&= 0 + \int_{-1}^1 \sqrt{1-t^2}dt = \pi/2 = A(-1)
\end{aligned}
$$

Similarly, one shows that $\lim_{x \to 1} A(x) = 0 = A(1)$. Thus, $A$ is continuous in $[-1, 1]$.

Now, the denominator of the derivative—$2\sqrt{1-x^2}$—is always positive, which means the derivative is always negative. Hence, the function $A$ is strictly decreasing in $]-1,1[$. As it is continuous in $[-1,1]$, this means $A$ decreases from $A(-1)=\pi/2$ to $A(1)=0$. From the Intermediate Value Theorem it now follows that for any $y\in[0,\pi/2]$ there exists (at least one) $x\in[-1,1]$ verifying $A(x)=y$ (note this means $A$ is surjective). The uniqueness of $x$ follows from the fact that $A$ is strictly monotonic (in this case, decreasing), and thus injective. Hence, definition 2.1 is proper.

**Remark 2.2.** Definition 2.1 already allows to think of cosine and sine as functions which argument is not an angle, but a real number. However, to avoid confusion with the Cartesian coordinates $x$ and $y$, in this section we shall continue to use Greek letters for the argument of sin and cos—dropping this practice only on the next section, §3. △

**Continuity of** sin **and** cos**.** We again begin with cosine. Let $\alpha\in[0,\pi]$. Per definition 2.1, $\cos\alpha$ is the only value for which $A(\cos\alpha)=\alpha/2$ holds. To prove continuity, first observe that $\lim_{\theta\to\alpha}A(\cos\theta)=\lim_{\theta\to\alpha}\theta/2=\alpha/2$. Now let $s\in[-1,1]$ be the only value such that $A(s)=\alpha/2$ (note this means $\cos\alpha=s$). Because $A$ is a continuous bijection in $[-1,1]$, we have $\lim_{x\to s}A(x)=A(s)=\alpha/2$, and for $t\neq s$, $\lim_{x\to t}A(x)=A(t)\neq A(s)$. Hence, we must have $\lim_{\theta\to\alpha}\cos\theta=s$, and as $\cos\alpha=s$, we conclude that cos is continuous in $[0,\pi]$.

The continuity of sine is now immediate, because given $\alpha$ as above, we have:

$$\lim_{\theta\to\alpha}\sin\theta=\lim_{\theta\to\alpha}\sqrt{1-\cos^2\theta}=\sqrt{1-\left(\lim_{\theta\to\alpha}\cos\theta\right)^2}=\sqrt{1-\cos^2\alpha}=\sin\alpha$$

**From** $[0,\pi]$ **to** $\mathbb{R}$**.** Having shown that sine and cosine are continuous on $[0,\pi]$, we first extend them to $[\pi,2\pi]$. If $\theta\in[\pi,2\pi]$, then by sheer arithmetic we have $-\theta\in[-2\pi,-\pi]$—and thus $(2\pi-\theta)\in[0,\pi]$. Moreover, if $\theta\in[\pi,2\pi]$, then we can write $\theta=\pi+\theta'$, where $\theta'\in[0,\pi]$. Then, we have $2\pi-\theta=2\pi-(\pi+\theta')=\pi-\theta'$. Going back momentarily to thinking about sines and cosines as Cartesian coordinates of a point one the unit circle, we expect that $\cos(\pi+\theta')=\cos(\pi-\theta')$ and $\sin(\pi+\theta')=-\sin(\pi-\theta')$—cf. figure 3.

Hence, for $\theta\in[\pi,2\pi]$ we have:

$$\begin{cases}\cos\theta=\cos(\pi+\theta')=\cos(\pi-\theta')=\cos(2\pi-\theta)\\ \sin\theta=\sin(\pi+\theta')=-\sin(\pi-\theta')=-\sin(2\pi-\theta)\end{cases}$$

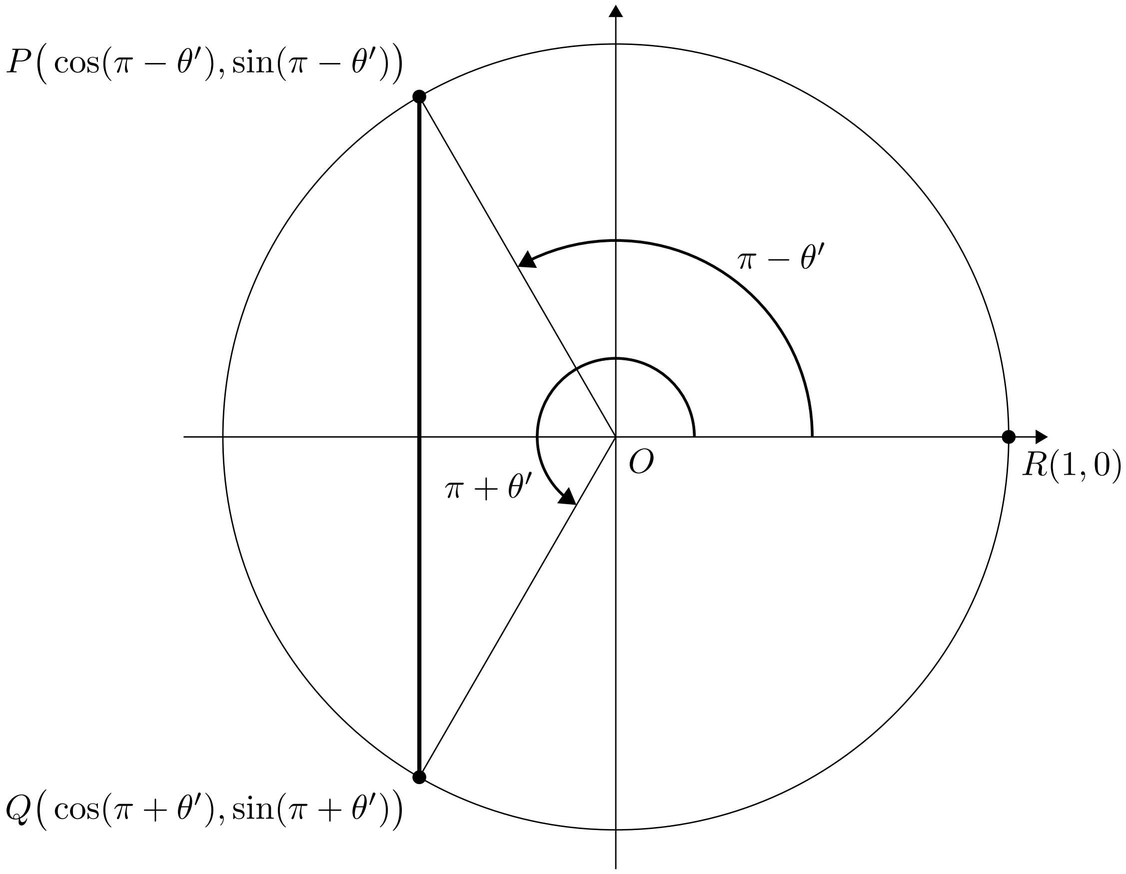


**Figure 3:** Given $\theta' \in [0, \pi]$, we depict the sine and cosine of $\pi - \theta'$ and $\pi + \theta$. In the specific case depicted, $\theta' \in [0, \pi/2]$. In the case of $\theta' \in [\pi/2, \pi]$, the line segment PQ would intersect the abscissae axis on a point to the right of the origin O.

And thus we can define, for $\theta \in [\pi, 2\pi]$:

$$\begin{cases} \cos\theta \overset{\text{def}}{=} \cos(2\pi - \theta) \\ \sin\theta \overset{\text{def}}{=} -\sin(2\pi - \theta) \end{cases} \tag{2.3}$$

This takes care of sin and cos for values in $[0, 2\pi]$. Note that it is of no importance that the intervals overlap on point $\pi$, because when $\theta = \pi$, $\theta = 2\pi - \theta$. To extend them from $[0, 2\pi]$ to all of $\mathbb{R}$ is easy:

$$\begin{cases} \cos(\theta + 2k\pi) \overset{\text{def}}{=} \cos\theta \\ \sin(\theta + 2k\pi) \overset{\text{def}}{=} \sin\theta \end{cases} \tag{2.4}$$

for $\theta \in [0, 2\pi]$ and $k \in \mathbb{Z}$.[6] It is immediate that both sine and cosine functions thus defined are periodic, with period $2\pi$.[7] Moreover, as we have $\cos 0 = \cos 2\pi = 1$ and $\sin 0 = \sin 2\pi = 0$, (2.4) implies that these newly defined sin and cos functions are continuous over all of $\mathbb{R}$.

**Derivatives of** sin **and** cos**.** We now come to the final task in this section, computing the derivatives of sine and cosine; we begin with the latter. Recall our original definition, where angles are restricted to $[0, \pi]$, and $\cos\theta$ is the unique number in $[-1, 1]$ for which it holds that $A(\cos\theta) = \theta/2$. We have seen that $A$ is a bijection between $[-1, 1]$ and $[0, \pi/2]$, and thus we can establish its inverse function, $A^{-1}$. Moreover, we have $\cos\theta = A^{-1}(\theta/2)$, and so we can compute $\cos'$ using $A'$ (computed above) and the Derivative Of Inverse Rule. This rule can only be applied if $A^{-1}$ is continuous on $[0, \pi/2]$—but as $A$ is a continuous strictly decreasing bijection (see above), we can apply the following theorem to conclude $A^{-1}$ is also a continuous bijective function:

**Theorem 2.5 (Continuity of the inverse).** *Let $a, b \in \mathbb{R}$ be two reals, with $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a continuous and strictly decreasing function. Then $f$ is a bijection from $[a, b]$ to $[f(b), f(a)]$, and its inverse, $f^{-1}$ is also strictly decreasing, and continuous.*

**Proof.** See page 13 in appendix A. ■

Applying now the Derivative Of Inverse Rule yields:

$$\cos'\theta = \left(A^{-1}(\theta/2)\right)' = (A^{-1})'(\theta/2) \cdot (1/2) = \frac{1}{A'\left[A^{-1}(\theta/2)\right]} \cdot \frac{1}{2}$$
$$= \frac{1}{A'(\cos\theta)} \cdot \frac{1}{2} = -\sqrt{1 - \cos^2\theta} = -\sin\theta$$

As for the derivative of the sine function, we have:

$$\sin'\theta = \left(\sqrt{1-\cos^2\theta}\right)' = \frac{1}{2}\frac{-2\cos\theta\cos'\theta}{\sqrt{1-\cos^2\theta}} = \frac{\cos\theta\sin\theta}{\sin\theta} = \cos\theta$$

It is important to keep in mind that $A'(x)$ is defined only for $x \in ]-1, 1[$ (see above), and so the derivatives we just computed are valid only for $\theta \in ]0, \pi[$. Let us now extend these differentiation rules to values $\theta \in ]\pi, 2\pi[$. Taking (2.3) into account, we have:

- For sin: $\sin'\theta = -\sin'(2\pi - \theta) = -\cos(2\pi - \theta) \times (-1) = \cos(2\pi - \theta) = \cos\theta$.
- For cos: $\cos'\theta = \cos'(2\pi - \theta) = -\sin(2\pi - \theta) \times (-1) = \sin(2\pi - \theta) = -\sin\theta$.

So, for $\theta \in ]0, \pi[ \cup ]\pi, 2\pi[$, we have $\sin' = \cos$ and $\cos' = -\sin'$. And from (2.4) it follows easily that these rules also apply to any $\theta \in \mathbb{R}$, except multiples of $\pi$—for which, we require the following theorem:

**Theorem 2.6 (Limit of the derivative).** *Let $X$ be a subset of $\mathbb{R}$ and $a$ an interior point of $X$, and $f\colon X \to \mathbb{R}$ a function continuous in $X$ and differentiable in $X \setminus \{a\}$, but such that $\lim_{x\to a} f'(x) = l$. Then $f'(a) = l$.*

**Proof.** See page 14 in appendix A. ■

Now let $X$ be a nonempty interval, such that the only multiple of $\pi$ that is contained in $X$, is $\pi$ itself. Consider the sin function: it is continuous in $X$, and for all $\theta \in I \setminus \{\pi\}$, we have $\sin' \theta = \cos \theta$. But cos is continuous over all $\mathbb{R}$, which means in particular that $\lim_{\theta \to \pi} \cos \theta = \cos \pi = -1$. Theorem 2.6 now immediately gives $\sin' \pi = -1 = \cos \pi$. The same reasoning can be done for any multiple of $\pi$, which shows the sin function is differentiable over all $\mathbb{R}$—and its derivative is cos. The reasoning is analogous to show that $\cos'(k\pi) = -\sin(k\pi)$ ($k \in \mathbb{Z}$).

Thus, we have shown that for any $\theta \in \mathbb{R}$, we have $\sin' = \cos$ and $\cos' = -\sin'$.

# 3 Taylor Series Of Sine And Cosine

In this section we derive the expressions of both functions in terms of the so-called Taylor series (see below). We leverage the fact that both sine cosine are solutions for the differential equation $f'' = -f$, $f \colon \mathbb{R} \to \mathbb{R}$—indeed, we would expect for any linear combination of sines and cosines to be a solution: $(a \cos x + b \sin x)'' = (-a \sin x + b \cos x)' = -(a \cos x + b \sin x)$. But this says nothing about whether any other solutions exist, so we proceed with a more generic approach. We shall require the values of cosine and sine at $0$: $\cos 0 = 1$ and $\sin 0 = 0$—cf. definition 2.1 if required.

The first observation is that any solution $f$ has derivatives of any order, over all of its domain: indeed, if $n$ is even, $f^{(n)}$ alternates between $f$ and $-f$; if it is odd, $f^{(n)}$ alternates between $f'$ and $-f'$. Furthermore, $f$ is continuous over all of $\mathbb{R}$, because its derivative is defined over all of $\mathbb{R}$. The same reasoning shows that $f^{(n)}$ is also continuous on $\mathbb{R}$, not just for $n = 0$ (which is $f$), but for any other $n \geq 1$ as well. In particular, $f'$ is continuous on $\mathbb{R}$. Which entails that, if we take any $\varepsilon > 0$, and consider the interval $[-\varepsilon, \varepsilon]$, both $f$ and $f'$ are bounded on that interval.

We now enlist some “big guns” of mathematical analysis, namely Taylor polynomials with so-called Lagrange remainder. We have the following theorem.

**Proposition 3.1.** *Let $n \geq 0$ be an integer, $X$ an interval of $\mathbb{R}$, and $f \colon X \to \mathbb{R}$ a function with continuous derivatives in $X$, up order $n+1$, and $x_0$ an interior point of $X$. Then for any $x \in X$, there exists $c$ strictly between $x_0$ and $x$, such that:*

$$
\begin{aligned}
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \dots \\
+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}
\end{aligned}
$$

**Proof.** This is a restatement, in a different form, of Theorem 4 (Taylor’s theorem) in Spivak [3, §19]—to which the reader is referred to for a proof. ∎

The series $\sum_{i=0}^{n}\left(f^{(i)}(x_0)(x-x_0)^i\right)/i!$ is called **Taylor polynomial of degree** $n$, and is usually denoted by $S_n(x)$. When $n \to +\infty$, we obtain the **Taylor series**. It is immediate that:

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} = f(x) - S_n(x)$$

The left hand side is called the **Lagrange form of the Taylor remainder**. And we can now prove the result that we actually need.

**Proposition 3.2.** *If there exists* $k \geq 0$ *such that for any* $n \geq 0$ *we have* $\left|f^{(n)}(x)\right| \leq k$ *for all* $x$ *in a neighbourhood*[8] *of* $x_0$*, then* $f(x)$ *equals its Taylor series for any* $x$ *in that neighbourhood.*

**Proof.** See page 14 in appendix A. ∎

Now, returning to our reasoning where we left off, let $m = \max_{[-\varepsilon,\varepsilon]}|f(x)|$ and $m' = \max_{[-\varepsilon,\varepsilon]}|f'(x)|$. Thus for $M = \max\{m, m'\}$, we have $\left|f^{(n)}(x)\right| \leq M$, for all $n \geq 0$ and $x \in [-\varepsilon, \varepsilon]$. Hence $f^{(n)}$ is also bounded on the interval $]-\varepsilon, \varepsilon[$, which is a neighbourhood of $x = 0$, and making $k = M$, we can apply proposition 3.2 to conclude that any such function must coincide with its Taylor series on point $x = 0$. I.e., we must have:

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$$

But because of the relation between derivatives of different orders we saw above, we can rewrite this as:

$$f(x) = f(0) + f'(0)x - \frac{f(0)}{2!}x^2 - \frac{f'(0)}{3!}x^3 + \frac{f(0)}{4!}x^4 + \frac{f'(0)}{5!}x^5 + \ldots$$

If we separately group terms with even and odd exponents, we obtain:

$$f(x) = \sum_{n=0}^{+\infty}\left((-1)^n f(0)\frac{x^{2n}}{(2n)!} + (-1)^n f'(0)\frac{x^{2n+1}}{(2n+1)!}\right)$$

We can further split the summation like this:

$$f(x) = f(0)\sum_{n=0}^{+\infty}\left((-1)^n\frac{x^{2n}}{(2n)!}\right) + f'(0)\sum_{n=0}^{+\infty}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right) \tag{3.3}$$

This equality holds for $x \in\, ]-\varepsilon, \varepsilon[$, but as $\varepsilon$ was arbitrarily chosen, it holds for any $x \in \mathbb{R}$. All of which shows that any function $f\colon \mathbb{R} \to \mathbb{R}$ that is a solution for $f'' = -f$, can be written in the form (3.3). But note the following: we know that one such function is $f(x) = a\cos x + b\sin x$, where $a, b$ are arbitrary values, and moreover, $f(0) = a$ and $f'(0) = b$. Thus, the left and right summations in (3.3) are the Taylor expansions for cosine and sine, respectively (it is a simple exercise to apply the ratio test[9] to show that both series have an infinite radius of convergence)—and moreover, all solutions to $f'' = -f$ are of the form $f(x) = a\cos x + b\sin x$. The following result is now immediate.

**Proposition 3.4.** *The Taylor expansions of cosine and sine are:*

$$\cos x = \sum_{n=0}^{+\infty}\left((-1)^n \frac{x^{2n}}{(2n)!}\right) \quad \text{and} \quad \sin x = \sum_{n=0}^{+\infty}\left((-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)$$

It is thus shown that the Taylor expansions of sine and cosine—often used, in analysis courses, as the *rigorous definitions* of sine and cosine—do indeed correspond to the "geometrical" sine and cosine one usually learns in high-school trigonometry.[10]

# 4 Some Properties of Sine And Cosine

We have $(\cos^2 x + \sin^2 x)' = 2\sin x\cos x - 2\cos x\sin x = 0$, meaning that $\cos^2 x + \sin^2 x$ is a constant function; and as $\cos 0 + \sin 0 = 1$, we conclude $\cos^2 x + \sin^2 x = 1$, for all $x \in \mathbb{R}$.

The well-known formulae for the sine and cosine of sums are also easy to derive. Letting $c$ be a fixed value, we have $\sin'(x + c) = \cos(x + c)$ and $\cos'(x + c) = -\sin(x + c)$, which means $f(x) = \sin(x + c)$ is also a solution of $f'' = -f$—and hence, it can be written as $f(0)\cos x + f'(0)\sin x$, from where we conclude that for any $x, c \in \mathbb{R}$, we have:

$$\sin(x + c) = \sin x\cos c + \cos x\sin c$$

Differentiating both sides we obtain:

$$\cos(x + c) = \cos x\cos c - \sin x\sin c$$

Note that, even though they were deduce by fixing one of the parcels, these formulae apply to any two real numbers.

From definition 2.1, we have $\cos\pi = -1$ and $\sin\pi = 0$. Together with the above formulas for sines and cosines of sums, this yields:

$$\begin{aligned} \sin(x + \pi) &= \sin x\cos\pi + \cos x\sin\pi = -\sin x \\ \cos(x + \pi) &= \cos x\cos\pi - \sin x\sin\pi = -\cos x \end{aligned} \tag{4.1}$$

This coincides with the intuition that arises from the "sine and cosine as Cartesian coordinates" approach depicted in figure 3. And this affords us another way to show that both sine and cosine have period $2\pi$:[11]

$$\sin(x + 2\pi) = \sin((x + \pi) + \pi) = -\sin(x + \pi) = -(-\sin x) = \sin x$$
$$\cos(x + 2\pi) = \cos((x + \pi) + \pi) = -\cos(x + \pi) = -(-\cos x) = \cos x$$

But we can go beyond this, and show that $2\pi$ is the *smallest positive* (i.e., the fundamental) period. To this end, note that if $T$ is a period of function $f$—meaning $f(x + T) = f(x)$, for all $x$ in the domain of $f$—then every multiple of $T$ is also a period of $f$.[12] Now, we know that $2\pi$ is a period of sin and cos, thus the fundamental period, if different from $2\pi$, must be a divisor of $2\pi$—that is to say, it must be of the form $2\pi/n$, with $n > 1$ an integer. However, from (4.1), we know $\pi$ is not a period of either function, and so we must actually have $n > 2$. Let us tackle sine first, by supposing we have $\sin(x + 2\pi/n) = \sin x$. But for $x = 0$, this would entail $\sin(2\pi/n) = \sin 0 = 0$, which is impossible, as $\sin(2\pi/n) = 0$ only holds for $n = 2$, and we have already established that we must have $n > 2$. Thus, no real of the form $2\pi/n$, with $n > 2$ an integer, can be a period of sin—which means $2\pi$ is its smallest period. Now for cosine, suppose we have $\cos(x + 2\pi/n) = \cos x$. Again for $x = 0$, this entails $\cos(2\pi/n) = \cos 0 = 1$, which is only possible if $n = 1$. Hence, no real of the form $2\pi/n$, with $n > 2$ an integer, can be a period of cos—which again means $2\pi$ is its smallest period.

# 5 Conclusion: A Discussion Of $\pi$

The main goal of this text—to establish the equivalence between geometric and analytical definitions of sine and cosine—has been accomplished. But I thought to end this writing with a note on $\pi$: in §2 we implicitly rely on the fact that for a circle of radius $r$, its perimeter is given by $2\pi r$, and its area by $\pi r^2$. We then use this to determine the derivatives of sin and cos, and use these to, in §3, derive the Taylor polynomials for both functions. However, in the literature, when defining sin and cos in terms of Taylor polynomials, it is also customary to define $\pi$ in terms of these functions.[13] That approach obviously does not work for us, because we have defined both cos and sin taking for granted that $\pi$ was already defined.

Actually, that is not strictly true—what we relied on, was the fact that the ratio between the perimeter of a circle, and its diameter, is constant (i.e., it is a finite real number). This cannot be proved—it has to be taken as an axiom of Euclidean geometry.[14] If we denote that ratio by $\pi$, it is immediate that the perimeter of a circle of radius $r$ is $2\pi r$. And furthermore, that its area is $\pi r^2$ can be shown by computing the following integral: $\int_0^r 2\pi t \, dt = 2\pi \int_0^r t \, dt = 2\pi(r^2/2 - 0) = \pi r^2$. Essentially, we "add up" the perimeters of all inner circles, for all radii from $0$ to $r$. This suffices for the reasoning carried out in §2—but it does not help one to compute the actual value of $\pi$, because we haven't really defined it.

So, how to actually define $\pi$? There are many possibilities, but one that relates to the work done in §2, is to define $\pi$ as twice the area under the upper unit circumference—which can be computed as $\int_{-1}^{1} \sqrt{1-t^2}dt$. That is, we set $\pi \overset{\text{def}}{=} 2\int_{-1}^{1} \sqrt{1-t^2}dt$.

# A Proofs

The following lemma is needed for proving theorem 2.5.

**Lemma A.1.** *Let* $X$ *be a subset of* $\mathbb{R}$ *and* $f\colon X \longrightarrow \mathbb{R}$ *be a function. The function* $f$ *is continuous at point* $a \in X$ *if and only if for every sequence* $\{x_n\}$ *of points of* $X$ *that converges to* $a$*, we have* $f(x_n) \longrightarrow f(a)$.

**Proof.** Let $\{x_n\}$ be a sequence of points of $X$ such that $\{x_n\} \longrightarrow a$. By definition of convergent sequence, this means that ($\varepsilon \in \mathbb{R}$ and $n, p \in \mathbb{N}$):

$$\forall \varepsilon > 0 \; \exists p \; \forall n \;\; n \geq p \Rightarrow |x_n - a| < \varepsilon \tag{A.2}$$

($\longrightarrow$) We assume $f$ is continuous at $a$, and want to show that this implies $\{f(x_n)\} \longrightarrow f(a)$. By definition of continuity, the following holds:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in X \;\; 0 < |x - a| < \delta \Rightarrow |f(x) - a| < \varepsilon \tag{A.3}$$

The condition we want to show—$\{f(x_n)\} \longrightarrow f(a)$—translates too:

$$\forall \varepsilon > 0 \; \exists p \; \forall n \;\; n \geq p \Rightarrow |f(x_n) - f(a)| < \varepsilon \tag{A.4}$$

The first observation, is that for any $n$ such that $x_n = a$, (A.4) holds trivially, so we can assume $x_n \neq a$. For these, it follows from (A.3) that given any real $\varepsilon > 0$, there exists a real $\delta > 0$ such that $\forall x \in X$, $0 < |x_n - a| < \delta \Rightarrow |f(x_n) - f(a)| < \varepsilon$. And from (A.2) we know that for *that* $\delta$, there exists $p$ such that $n \geq p \Rightarrow |x_n - a| < \delta$. Combining these two statements yields (A.4).

($\longleftarrow$) We assume that $\{x_n\} \longrightarrow a$ implies $\{f(x_n)\} \longrightarrow f(a)$, for any arbitrary sequence $\{x_n\}$ of points of $X$. We want to show that this in turn implies that $f$ is continuous at $a$, i.e., that $\lim_{x \to a} f(x) = f(a)$. We will prove this via the contrapositive, that is, we will assume that $f$ is *not* continuous at $a$, and show that in such a case, there exists a sequence $\{x_n\}$ such that $\{x_n\} \longrightarrow a$ and $\{f(x_n)\} \not\longrightarrow f(a)$.

So, if $f$ is not continuous at $a$, this means that the following condition holds:

$$\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists x \in X \;\; 0 < |x - a| < \delta \wedge |f(x) - a| \geq \varepsilon \tag{A.5}$$

Fix an $\varepsilon$ for which (A.5) holds. Then, if we set $\delta = 1$, there exists at least one value of $x$ such that the sub-condition $0 < |x - a| < \delta \wedge |f(x) - a| \geq \varepsilon$ holds—let $x_1$ equal that value of $x$. More generally, let $x_n$ be a value of $x$ for which the same sub-condition holds when $\delta = 1/n$. It is clear that $\{x_n\} \to a$, but $\{f(x_n)\} \not\to f(a)$, which concludes the proof. ■

**Theorem 2.5 (Continuity of the inverse).** *Let* $a, b \in \mathbb{R}$ *be two reals, with* $a < b$*, and let* $f\colon [a, b] \to \mathbb{R}$ *be a continuous and strictly decreasing function. Then* $f$ *is a bijection from* $[a, b]$ *to* $[f(b), f(a)]$*, and its inverse,* $f^{-1}$ *is also strictly decreasing, and continuous.*

**Proof.** It is immediate that f being strictly monotone (decreasing in the present case), it is also injective. We must show that for any $x \in [a, b]$, $f(x) \in [f(b), f(a)]$. This is trivial for either $x = a$ or $x = b$; for $x \in ]a, b[$, f being strictly decreasing means that $a < x < b$ implies $f(a) > f(x) > f(b)$. From here it also follows that $[f(b), f(a)]$ is a nonempty interval.

To show that f is surjective in $[f(b), f(a)]$, we first note that as it is continuous in $[a, b]$, by the Intermediate Value Theorem, for any $y \in [f(b), f(a)]$, there exists $x \in [a, b]$ such that $f(x) = y$—and this $x$ is unique due to f being injective. Hence, f is bijective, and thus we can define $f^{-1}\colon [f(b), f(a)] \to [a, b]$.

To show that $f^{-1}$ is also strictly decreasing, let $y_1, y_2 \in [f(b), f(a)]$, with $y_1 < y_2$, and let $x_1 = f^{-1}(y_1) \Leftrightarrow y_1 = f(x_1)$ and $x_2 = f^{-1}(y_2) \Leftrightarrow y_2 = f(x_2)$. Reasoning by contradiction, assume that $f^{-1}(y_1) \leq f^{-1}(y_2)$. We have: $x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2) \Rightarrow y_1 \geq y_2$, which is a contradiction.[15] Thus we conclude that $f^{-1}(y_1) > f^{-1}(y_2)$.

Our last task is to show that $f^{-1}$ is continuous—and for this, we shall use lemma A.1 to show that for any sequence $\{y_n\}$ in $[f(b), f(a)]$ such that $\{y_n\} \to y_0$, we have $\left\{f^{-1}(y_n)\right\} \to f^{-1}(y_0)$. Let $x_0 = f^{-1}(y_0) \Leftrightarrow y_0 = f(x_0)$. As f is bijective, for any $y_n$ there exists an unique $x_n$ such that $y_n = f(x_n)$. Thus we can rewrite $\{y_n\} \to y_0$ as $\{f(x_n)\} \to f(x_0)$. Due to the continuity of f, **if** $\{x_n\}$ **is convergent**, it must converge to $x_0$—otherwise (i.e., if it converged to any other value, say $a \neq x_0$), by lemma A.1 and the unicity of limits we would have $f(x_0) = f(a)$, which is impossible as f is injective. To prove that $\{x_n\}$ is indeed convergent, we first obverse that for any $n$, $x_n \in [a, b]$, which is a *bounded* subset of $\mathbb{R}$. Hence, if $\{x_n\}$ is divergent, as neither it nor any of its subsequences can diverge to (positive or negative) infinity, it must be because there exist at least two subsequences that converge to different limits.[16] Let us then, suppose there exist two subsequences of $\{x_n\}$, $\{x_{n_i}\}$ and $\left\{x_{n_j}\right\}$, such that $\{x_{n_i}\} \to c_i$ and $\left\{x_{n_j}\right\} \to c_j$, with $c_i \neq c_j$. As f is continuous, $\{f(x_{n_i})\} \to f(c_i)$ and $\left\{f(x_{n_j})\right\} \to f(c_j)$. However, as both $\{x_{n_i}\}$ and $\left\{x_{n_j}\right\}$ are subsequences of $\{x_n\}$, $\{f(x_{n_i})\}$ and $\left\{f(x_{n_j})\right\}$ are subsequences of $\{f(x_n)\}$, which converges to $f(x_0)$—and thus, so must $\{f(x_{n_i})\}$ and $\left\{f(x_{n_j})\right\}$. This entails $f(c_i) = f(c_j) = f(x_0)$, and f being injective, we conclude $c_i = c_j$, which is a contradiction. Hence, $\{x_n\} \to x_0$. We now have $\left\{f^{-1}(y_n)\right\} = \left\{f^{-1}\left[f(x_n)\right]\right\} = \{x_n\} \to x_0 = f^{-1}(y_0)$, i.e., $\left\{f^{-1}(y_n)\right\} \to$

$f^{-1}(y_0)$. As $\{y_n\}$ was an arbitrary convergent sequence, by lemma A.1 we conclude that $f^{-1}$ is continuous on $[f(b), f(a)]$. ■

**Theorem 2.6 (Limit of the derivative).** *Let* $X$ *be a subset of* $\mathbb{R}$ *and* $a$ *an interior point of* $X$*, and* $f\colon X \to \mathbb{R}$ *a function continuous in* $X$ *and differentiable in* $X \setminus \{a\}$*, but such that* $\lim_{x\to a} f'(x) = l$*. Then* $f'(a) = l$*.*

**Proof.** We want to prove that:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in X \setminus \{a\}\ \ 0 < |x - a| < \delta \Rightarrow \left|\frac{f(x) - f(a)}{x - a} - l\right| < \varepsilon \qquad \text{(A.6)}$$

Now, applying Lagrange's theorem to $f$ in the closed interval with extrema $a$ and $x$, we know that there exists $c$ (dependent on $x$) such that:

$$\left|\frac{f(x) - f(a)}{x - a} - l\right| = |f'(c) - l| \qquad \text{(A.7)}$$

Observe also that $|c - a| < |x - a|$, because $c$ belongs to the closed interval with extrema $a$ and $x$. Now $\lim_{x\to a} f'(x) = l$ means that:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in X \setminus \{a\}\ \ 0 < |x - a| < \delta \Rightarrow |f'(x) - l| < \varepsilon \qquad \text{(A.8)}$$

Thus, given any $\varepsilon > 0$, we chose $\delta > 0$ so as to verify the implication in (A.8), and as for any $x$ verifying the antecedent, the corresponding $c$ of (A.7) verifies $|c - a| < |x - a| < \delta$, we have $|f'(c) - l| < \varepsilon$. It is now immediate that (A.6) holds. ■

**Proposition 3.2.** *If there exists* $k \geq 0$ *such that for any* $n \geq 0$ *we have* $\left|f^{(n)}(x)\right| \leq k$ *for all* $x$ *in a neighbourhood of* $x_0$*, then* $f(x)$ *equals its Taylor series for any* $x$ *in that neighbourhood.*

**Proof.** Let $X$ be a neighbourhood of $x_0$ verifying the required conditions, and let $x \in X$. Applying proposition 3.1 to $f$, and setting $S_n(x) = \sum_{i=0}^{n} \left(f^{(i)}(x_0)(x - x_0)^i\right)/i!$, we have:

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} = f(x) - S_n(x)$$

for some $c$ strictly between $x$ and $x_0$—and thus, $c \in X$. We have:

$$|f(x) - S_n(x)| = \left|f^{(n+1)}(c)\right| \cdot \frac{\left|(x - x_0)^{n+1}\right|}{(n+1)!} \leq k\frac{|(x - x_0)|^{n+1}}{(n+1)!}$$

Now by the ratio test,[17] one sees that $\sum y^n/n!$ is convergent, for any $y \in \mathbb{R}$—and so, $y^n/n! \to 0$ when $n \to +\infty$. Hence, $k(|(x - x_0)|^{n+1})/((n+1)!) \to 0$, entailing $f(x) - S_n(x) \to 0$ for every $x$ in a neighbourhood of $x_0$, and thus, $f$ equals its Taylor series in that neighbourhood. ■

# Notes

1. For an example of this approach, see Tao [5, §4.7]. Being defined for any value in $\mathbb{C}$, they are of course defined for all of $\mathbb{R}$.
2. Cf. Tao, loc. cit.
3. As is conventional, counter-clockwise angles are considered **positive**, and clockwise angles are considered **negative**.
4. Recall that if $u(x)$ is a function, then $(\sqrt{u(x)})' = u'(x)/(2\sqrt{u(x)})$.
5. See, e.g., Tao [4, Prop. 10.1.10, in §10].
6. And so, to compute, say, $\cos\gamma$, for any $\gamma \in \mathbb{R}$, compute the integer $k$ as $k = \lfloor \gamma/2\pi \rfloor$, where $\lfloor x \rfloor$ is the largest integer not greater than the real $x$. Then we have $\gamma = 2k\pi + \theta$, for some $\theta \in [0, 2\pi]$. And hence, per (2.4), $\cos\gamma = \cos\theta$.
7. One can now ask whether or not $2\pi$ is the *smallest* period. It is, as we will show at the end of §4.
8. A **neighbourhood** of a point $a$ is an open interval of the form $]a - \varepsilon, a + \varepsilon[$, for some real number $\varepsilon > 0$.
9. See for example, Tao [4, §7.5].
10. As a "sanity check," it is immediate to verify that according to their respective Taylor expansions, we have $\cos 0 = 1$ and $\sin 0 = 0$, just as expected from definition 2.1. Also, power series—of which the Taylor series is a particular case—are indefinitely differentiable, and so term-wise differentiation shows that $\sin' x = \cos x$ and $\cos' x = -\sin x$.
11. Cf. (2.4).
12. This easy to see by setting $x = x + T$ in $f(x + T) = f(x)$.
13. Tao [5, §4.7], for example, defines $\pi$ as the smallest zero sin in $]0, +\infty[$.
14. In fact, in other, so-called *non-Euclidean* geometries, that ratio can be different for different circles! For example, in spherical geometry, you can have '$\pi$'$= 2$! See the answer of user SRM (April 21st, 2014), on this `Math StackExchange` thread [1].
15. The step $x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$ might need additional justification. Rewrite it as $(x_1 < x_2 \vee x_1 = x_2) \Rightarrow (f(x_1) > f(x_2) \vee y_1 = y_2)$, and recall that as $f$ is strictly decreasing, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$. The original assertion now follows from the following property of propositional logic: given propositions $a, b, c, c'$, with $c \Leftrightarrow c'$, $a \Rightarrow b$ is true if and only if $(a \vee c) \Rightarrow (b \vee c')$ is true.

   To see why this equivalence holds, recall that $a \Rightarrow b$ is false only when $a$ is true and $b$ is false. It straightforward to check that this is also the only way to make $(a \vee c) \Rightarrow (b \vee c')$ false.
16. Recall that by the Bolzano-Weierstraß theorem any bounded sequence has a convergent subsequence.
17. Cf. note 9 above.

# References


1. MATH STACKEXCHANGE, 2014. *Is there any geometry where ratio of circles's circumference to its diameter is rational?*. See https://math.stackexchange.com/questions/762423/is-there-any-geometry-where-ratio-of-circles-circumference-to-its-diameter-is-r, last access on July 30, 2023. Cited on p. 15.
2. **Sarrico**, Carlos, 1997. *Análise Matemática*. Lisboa: Gradiva. In Portuguese. Cited on p. 2.
3. **Spivak**, Michael, 1976. *Calculus*. W. A. Benjamin, Inc. Cited on pp. 2 and 8.
4. **Tao**, Terence, 2022 [2006]. *Analysis I*. New Delhi: Hindustan Book Agency, 4th edition. Cited on p. 15.
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