Sine And Cosine in $\mathbb{R}$: From Geometric Definitions To Taylor Formulae

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Abstract. This paper bridges the gap between the "geometric definitions" of sine and cosine, based on (tri)angles and/or the trigonometric circle, and what is usually presented in, e.g., undergraduate mathematics as their rigorous definitions, based on Taylor polynomials and/or the exponential function.

Prerequisites. The reader is expected to be comfortable with topics usually taught in freshman calculus courses, viz., sequences, (one variable) functions, and taking limits of both, continuity, differentiation and power/Taylor series.

Keywords: trigonometry, sine, cosine, differentiation, Taylor series.

1 Introduction

Most people learn trigonometry in the context of geometry: angles and triangles, and the trigonometric circle, and define sine and cosine using these concepts. The problem with this approach, is that it is not rigorous. Conversely, rigorous analytical definitions of these functions are often anything but intuitive: typically, one either starts from the fact that for sin and cos, we have $f'' = -f$, and then derives their Taylor expansion, which is then used *as the definition* sin and cos; or one first defines the complex exponential function, and then defines sine and cosine in terms of that exponential.[1] The problem is that it is then well-nigh impossible to see how the sine and cosine functions defined in this manner coincide with their geometrical counterparts. To be sure, one can prove that these analytical definitions have some of the same properties as the geometrical sine and cosine,[2] but this does not prove that the analytically defined functions actually describe the original geometrical "reality."

   Such a proof is not easily found on the literature—I could only find a proof sketch from Spivak, cf. the "Credits" paragraph, below—and hence

I purport to provide one here. I shall stick with sin and cos only, because all other trigonometric functions can be derived from them.

Strategy. We will begin by defining the cosine function, in a manner that agrees with our geometric understanding of that function, but made rigorous by the use of the integral. This will be done, at first, for angles on a limited interval, but the definition will then be broadened to any real number. Having defined cosine, defining sine from it is straightforward. These functions will then be proved to be continuous, and indefinitely differentiable. This will be §2.

After that is done, we will leverage $\sin'$ and $\cos'$ to derive the Taylor series for each function. This will show that the formulae that in calculus courses is usually taken to be the *definition* of sine and cosine, do indeed coincide with the "geometric sine" and "geometric cosine" that one usually learns in high-school trigonometry. This will be §3.

Following that, we will derive, taking the Taylor polynomials as the starting point, both some expected properties—e.g., $\sin^2 x + \cos^2 x = 1$— as well as some well-known formulas, such as the sine and cosine of sums, without any appeal to geometric intuition. This section, §4, will end with an analysis of the periodicity of the sine and cosine functions.

The paper concludes with (one way of) formally defining the number π, in §5. In the appendices, first we relate the trigonometric circle to triangles (§A), followed by some brief generalities about periodic functions (§B; some of the results here justify the computations done at the end of §4), and after that, the proofs for some theorems used (without proof) throughout the text are given (§C).

Credits. The idea of defining cosine in terms of the integral is largely drawn from Spivak [3, §15], although we provide some of the details absent therein. The presentation of the Taylor polynomials takes some inspiration from Sarrico [2, §9].

2 From Geometry To Formal Definitions

We will start with cosine, because once that is defined, it is trivial to define sine from it. The idea is to go from thinking of cosine as a function that applies to an *angle,* to thinking of it as a function that applies to *an arbitrary real number*. So starting with angles, if one measures them counter-clockwise[3] from the positive semi-axis of the abscissae, then one can identity the amplitude of an angle with the corresponding length of the arc of the unit circumference delimited by that angle—which is, of course, how one defines *radians*. In particular, as the whole unit circumference has length 2π, it corresponds to an amplitude of 2π radians. Also,
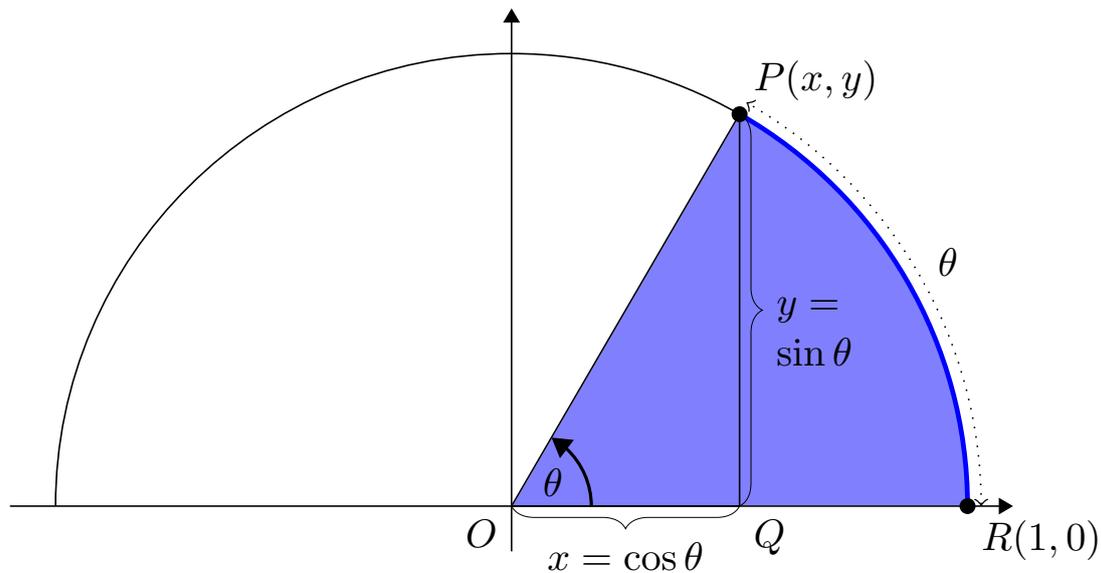
Figure 1: Sine and cosine on the trigonometric circle. Note that $y = \sqrt{1 - x^2}$.

when the sector is the entire unit circle, its area is π, i.e., half of the length of the corresponding arc (the entire unit circumference).

Accordingly, given an angle θ as in figure 1, the corresponding arc has length θ (in blue), and the corresponding sector (purple-ish shade) has area $\theta/2$. The reader likely learned that $\cos\theta$ and $\sin\theta$ are defined in such a way as to make $(\cos\theta, \sin\theta)$ be the coordinates of point P (in the same figure)—i.e., $(x, y) = (\cos\theta, \sin\theta)$. Towards a more formal approach, begin by observing that, for any x in $[-1, 1]$, there exists one, and only one point P that lies on the upper unit circle, and has x as its abscissae coordinate—namely, $P = (x, \sqrt{1 - x^2})$. Thus, if we had a continuous, bijective function $L\colon [-1, 1] \to [0, \pi]$, such that $L(x)$ is the length of the arc PR as depicted in figure 1, we could define $\cos\theta$ as the only value x such that $L(x) = \theta$ (for $\theta \in [0, \pi]$). This requires integration, but one usually learns it by reasoning over areas, rather than lengths. Thus, we will instead construct a (continuous and bijective) function $A\colon [-1, 1] \to [0, \pi/2]$, such that $A(x)$ expresses the *area of the sector* corresponding to the arc PR (i.e., the "slice" corresponding to angle $P\hat{O}R$ in figure 1) in terms of x—and then, $\cos\theta$ will be the only x such that $A(x) = \theta/2$. This, as mentioned, is only valid for $\theta \in [0, \pi]$. However, throughout the current section, this restriction will be gradually removed, so that in the end we will have cosine (and sine) defined for any arbitrary real value.

We define function A as follows:

$$A(x) = \frac{x\sqrt{1 - x^2}}{2} + \int_x^1 \sqrt{1 - t^2}\,dt \qquad (2.1)$$

To see that it indeed computes the desired area, observe that the left parcel computes the area of right triangle PQO in figure 1, and the integral
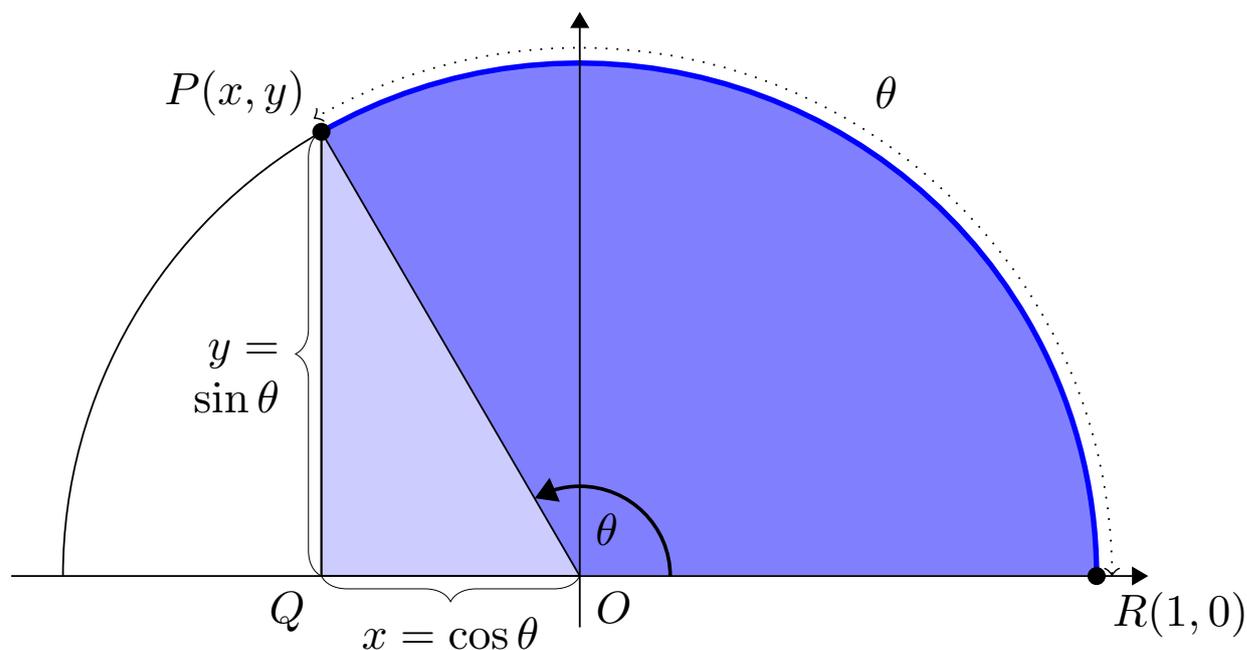
Figure 2: When x is negative.

computes the remaining area of that sector, that is "above" line segment $\overline{QR}$. Observe that this formula also holds for negative values of x (i.e., when $\theta > \pi/2$), because then the left parcel is negative, and corresponds exactly to the area that has to be *subtracted* from the integral to obtain the area of the unit circle sector—cf. the triangle PQO in figure 2.

To show that it is a bijection, we begin by showing it is continuous in $[-1, 1]$. Continuity inside the interval is shown by differentiation (in fact, the derivative is only defined for $x \in\,]-1, 1[$, because the derivative[4] of $\sqrt{x}$ is not defined at $x = 0$—hence, that of $\sqrt{1-x^2}$ is not defined for $x = \pm 1$). Using the usual differentiation rules, together with the Fundamental Theorem of Calculus, we have:

$$
\begin{aligned}
A'(x) &= \frac{1}{2}\left(x \cdot \frac{-2x}{2\sqrt{1-x^2}} + \sqrt{1-x^2} \right) - \sqrt{1-x^2} \\
&= \frac{1}{2}\left(\frac{-x^2 + (1-x^2)}{\sqrt{1-x^2}} \right) - \sqrt{1-x^2} \\
&= \frac{1-2x^2}{2\sqrt{1-x^2}} - \sqrt{1-x^2} \\
&= \frac{1-2x^2 - 2(1-x^2)}{2\sqrt{1-x^2}} = \frac{-1}{2\sqrt{1-x^2}}
\end{aligned}
\tag{2.2}
$$

From a well-known theorem in calculus, the fact that A is differentiable in $]-1, 1[$ means it is also continuous on that interval[5]—but we can also prove continuity at the extremes. Recall that by the Fundamental Theorem Calculus, we know that the integral in the definition of A (2.1) is

continuous in $[-1, 1]$. We have:

$$\lim_{x \to -1} A(x) = \lim_{x \to -1} \frac{x\sqrt{1-x^2}}{2} + \lim_{x \to -1} \int_x^1 \sqrt{1-t^2}dt$$

$$= 0 + \int_{-1}^1 \sqrt{1-t^2}dt = \pi/2 = A(-1)$$

Similarly, one shows that $\lim_{x \to 1} A(x) = 0 = A(1)$. Thus, A is continuous in $[-1, 1]$.

That A is bijective follows from the fact that it is strictly decreasing: the denominator of the derivative—$2\sqrt{1-x^2}$—is always positive, which means the derivative is always negative. Hence, the function A is strictly decreasing in $]-1, 1[$. As it is continuous in $[-1, 1]$, this means A (strictly) decreases from $A(-1) = \pi/2$ to $A(1) = 0$. Thus, we can define the inverse function, $A^{-1} \colon [0, \pi/2] \to [-1, 1]$, which the next theorem shows will also be a continuous and strictly decreasing bijection:

Theorem 2.3 (Continuity of the inverse). *Let $a, b \in \mathbb{R}$ be two reals, with $a < b$, and let $f \colon [a, b] \to \mathbb{R}$ be a continuous and strictly decreasing function. Then f is a bijection from $[a, b]$ to $[f(b), f(a)]$, and its inverse, f^{-1} is also strictly decreasing, and continuous.*

Proof. See page 19 in appendix C.   ∎

Defining the cosine of an angle θ to be the value x such that $A(x) = \theta/2$, is now proper: it just means that $\cos \theta = A^{-1}(\theta/2)$.

Definition 2.4. *Let $\theta \in [0, \pi]$. $\cos \theta$ is the (unique) value $a \in [-1, 1]$ such that $A(a) = \theta/2$. And $\sin \theta \overset{\text{def}}{=} \sqrt{1 - \cos^2 \theta}$.*

Remark 2.5. After reading definition 2.4, the reader may wonder why we don't define $\cos \theta$ simply as $A^{-1}(\theta/2)$. The reason is because we do not have an explicit formula for A^{-1}—and so, to be able to compute the cosine of given angles using the definition (see below), we rely on function A, for which we have an explicit formula, viz. (2.1).   △

Remark 2.6. Because A is a bijection from $[-1, 1]$ to $[0, \pi/2]$, given any $t \in [-1, 1]$, there is one, and only one $\theta \in [0, \pi]$ such that $\cos \theta = t$—namely, $\theta = 2A(t)$.   △

Remark 2.7. Definition 2.4 already allows us to think of cosine and sine as functions which take as an argument not an angle, but a real number. However, to avoid confusion with the Cartesian coordinates x and y, in this section we shall continue to use Greek letters for the argument of sin and cos—dropping this practice only on the next section, §3.   △

Extending sin **and** cos **from** $[0, \pi]$ **to** $\mathbb{R}$**.** We first extend them to $[\pi, 2\pi]$. If $\theta \in [\pi, 2\pi]$, then by sheer arithmetic we have $-\theta \in [-2\pi, -\pi]$—and thus $(2\pi - \theta) \in [0, \pi]$. Moreover, if $\theta \in [\pi, 2\pi]$, then we can write $\theta = \pi + \theta'$, where $\theta' \in [0, \pi]$. Then, we have $2\pi - \theta = 2\pi - (\pi + \theta') = \pi - \theta'$. Going back momentarily to thinking about sines and cosines as Cartesian coordinates of a point one the unit circle, we expect that $\cos(\pi + \theta') = \cos(\pi - \theta')$ and $\sin(\pi + \theta') = -\sin(\pi - \theta')$—cf. figure 3.

Hence, for $\theta \in [\pi, 2\pi]$ and $\theta' \in [0, \pi]$, we have:

$$\begin{cases} \cos \theta = \cos(\pi + \theta') = \cos(\pi - \theta') = \cos(2\pi - \theta) \\ \sin \theta = \sin(\pi + \theta') = -\sin(\pi - \theta') = -\sin(2\pi - \theta) \end{cases}$$

And thus we can define, for $\theta \in [\pi, 2\pi]$:

$$\begin{cases} \cos \theta \overset{\text{def}}{=} \cos(2\pi - \theta) \\ \sin \theta \overset{\text{def}}{=} -\sin(2\pi - \theta) \end{cases} \tag{2.8}$$

This takes care of sin and cos for values in $[0, 2\pi]$. Note that it is of no importance that the intervals overlap on point π, because when $\theta = \pi$, $\theta = 2\pi - \theta$.

Remark 2.9. We can recast the definition of $\cos \theta$ for $\theta \in [\pi, 2\pi]$ using the A function, *a la* definition 2.4: $\cos \theta$ is the unique value $a \in [-1, 1]$ such that $A(a) = (2\pi - \theta)/2$. This is because $2\pi - \theta$ is a bijection from $[\pi, 2\pi]$ to $[0, \pi]$.

And remark 2.6 can be similarly updated: given any $t \in [-1, 1]$, there is one, and only one $\theta \in [\pi, 2\pi]$ such that $\cos \theta = t$—namely, $\theta = 2\pi - 2A(t)$. This is because as A is a bijection (from $[-1, 1]$ to $[0, \pi/2]$), then $2\pi - 2A(x)$ is also a bijection (from $[-1, 1]$ to $[\pi, 2\pi]$). $\triangle$

To extend sine and cosine from $[0, 2\pi]$ to all of $\mathbb{R}$ is easy:

$$\begin{cases} \cos(\theta + 2k\pi) \overset{\text{def}}{=} \cos \theta \\ \sin(\theta + 2k\pi) \overset{\text{def}}{=} \sin \theta \end{cases} \tag{2.10}$$

for $\theta \in [0, 2\pi[$ and $k \in \mathbb{Z}$. This effectively replicates the behaviour of sin and cos in $[0, 2\pi[$, to all of $\mathbb{R}$. Why do we exclude the possibility of having $\theta = 2\pi$? For two reasons. First, (2.10) would not work if, for example, $\cos 0 \neq \cos 2\pi$: indeed, we could set $\theta = 2\pi$ and obtain, with $k = -1$, $\cos(2\pi + 2(-1)\pi) = \cos 2\pi \Leftrightarrow \cos 0 = \cos 2\pi$, which would be contradictory. However, we do have $\cos 0 = \cos 2\pi$—and indeed, the same holds for sine (see below)—which means we could allow setting $\theta = 2\pi$. The reason we don't, is the following: given any $\gamma \in \mathbb{R}$, we can write it as $\theta + 2\pi\lfloor \gamma/2\pi \rfloor$, with $\theta \in [0, 2\pi[$ and $k = \lfloor \gamma/2\pi \rfloor$, and this decomposition is unique.[6] However, if we could set $\theta = 2\pi$, then any
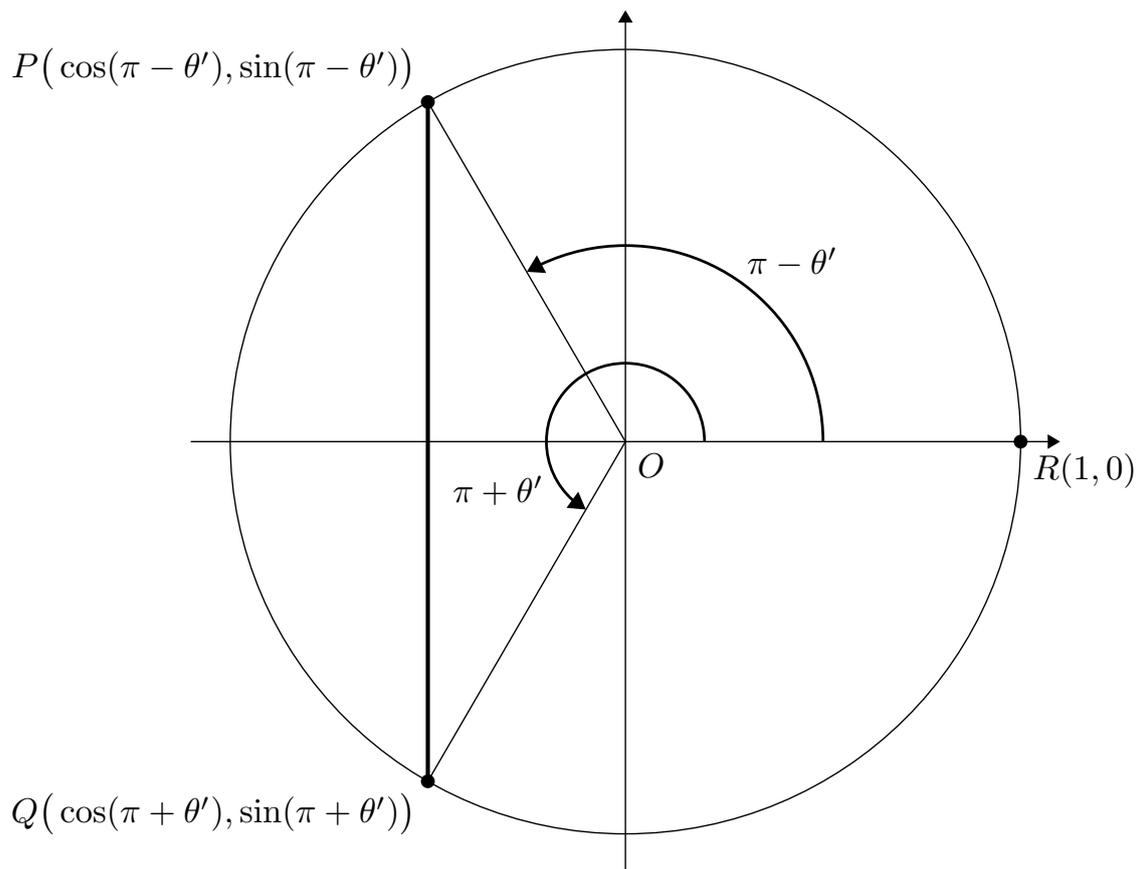
Figure 3: Given $\theta' \in [0, \pi]$, we depict the sine and cosine of $\pi - \theta'$ and $\pi + \theta'$. In the specific case depicted, $\theta' \in [0, \pi/2]$. In the case of $\theta' \in [\pi/2, \pi]$, the line segment $\overline{PQ}$ would intersect the abscissae axis on a point to the right of the origin O.

$\gamma = 2\pi s$, with s an integer, could be written either setting $\theta = 0$ and $k = s$, or $\theta = 2\pi$ and $k = s - 1$. This redundancy is inelegant, and thus we disallow setting $\theta = 2\pi$.

In light of the above, to compute, say, $\sin \gamma$, we simply write $\gamma = \theta + 2\pi \lfloor \gamma/2\pi \rfloor$ as above—from whence, per (2.10), follows $\sin \gamma = \sin \theta$.

Some "standard" values of sin and cos. We will now illustrate computing values of sine and cosine, from their formal definitions.[7] To compute $\cos 0$, we require a value $x \in [-1, 1]$ such that $A(x) = 0/2 = 0$. As (2.1) makes clear, $A(1) = 0$—which means $\cos 0 = 1$. For $\cos 2\pi$, by (2.8) we have $\cos 2\pi = \cos(2\pi - 2\pi) = \cos 0 = 1$. Conversely, when we restrict ourselves to interval $[0, \pi]$, by remark 2.6 $\cos \theta = 1 \Leftrightarrow \theta = 2 \times A(1) = 2 \times 0 \Leftrightarrow \theta = 0$. Now suppose $\theta \in [\pi, 2\pi]$. By remark 2.9 we have $\cos \theta = 1 \Leftrightarrow \theta = 2\pi - 2A(1) = 2\pi - 0 = 2\pi$. Thus, we have shown that for $\theta \in [0, 2\pi]$, $\cos \theta = 1$ if and only if $\theta = 0 \vee \theta = 2\pi$.

Next, for $\cos \pi/2$, we search an $x \in [-1, 1]$ such that $A(x) = (\pi/2)/2 = \pi/4$—and $x = 0$ fits the bill, which means $\cos \pi/2 = 0$. For $\cos 3\pi/2$, via (2.8) comes $\cos 3\pi/2 = \cos(2\pi - 3\pi/2) = \cos \pi/2 = 0$. Conversely, $\cos \theta = 0 \Leftrightarrow \theta = 2A(0) = \pi/2$, in $[0, \pi]$. And for $\theta \in [\pi, 2\pi]$, we have $\theta = 2\pi - 2A(0) = 2\pi - \pi/2 = 3\pi/2$. All of which shows that for

$\theta \in [0, 2\pi]$, $\cos\theta = 0 \Leftrightarrow \theta = \pi/2 \vee \theta = 3\pi/2$.

Now for sine, it follows easily that $\sin 0 = \sqrt{1 - \cos^2 0} = \sqrt{1 - (1)^2} = 0$. For $\sin\pi$, just as we showed above that in $[0, 2\pi]$, $\cos\theta = 1$ if and only if $\theta = 0 \vee \theta = 2\pi$, and via a similar reasoning one shows that $\cos\theta = -1$ if and only if $\theta = \pi$. And then $\sin\pi = \sqrt{1 - \cos^2\pi} = \sqrt{1 - (-1)^2} = 0$. For $\sin 2\pi$ comes $\sin 2\pi = \sin(2\pi - 2\pi) = \sin 0 = 0$. Conversely, we want to solve $\sin\theta = 0$. For $\theta \in [0, \pi]$, we have $\sin\theta = 0 \Leftrightarrow \sqrt{1 - \cos^2\theta} = 0 \Leftrightarrow \cos^2\theta = 1 \Leftrightarrow \cos\theta = 1 \vee \cos\theta = -1$. By our results above for cosine, this means that $\theta = 0 \vee \theta = \pi$. For $\theta \in [\pi, 2\pi]$, we have $\sin\theta = 0 \Leftrightarrow -\sin(2\pi - \theta) = 0 \Leftrightarrow \sqrt{1 - \cos^2(2\pi - \theta)} = 0 \Leftrightarrow \cos(2\pi - \theta) = 1 \vee \cos(2\pi - \theta) = -1$. As $\theta \in [\pi, 2\pi] \Leftrightarrow (2\pi - \theta) \in [0, \pi]$, by the same above results for cosine, this means that $2\pi - \theta = 0 \vee 2\pi - \theta = \pi \Leftrightarrow \theta = 2\pi \vee \theta = \pi$. Combining all this, we conclude that for $\theta \in [0, 2\pi]$, $\sin\theta = 0$ if and only if $\theta = 0 \vee \theta = \pi \vee \theta = 2\pi$.

Next, we take $\sin\pi/2$, which by definition is $\sqrt{1 - \cos^2(\pi/2)} = \sqrt{1 - 0} = 1$. Conversely, $\sin\theta = 1 \Leftrightarrow \sqrt{1 - \cos^2\theta} = 1 \Leftrightarrow \cos^2\theta = 0 \Leftrightarrow \cos\theta = 0$, which as we saw above, has only one solution in $[0, \pi]$, namely $\theta = \pi/2$. For interval $\theta \in [\pi, 2\pi]$, we have $\sin\theta = 1 \Leftrightarrow -\sin(2\pi - \theta) = 1 \Leftrightarrow \sin(2\pi - \theta) = -1 \Leftrightarrow \sqrt{1 - \cos^2(2\pi - \theta)} = -1$, which is impossible—meaning there is no $\theta \in [\pi, 2\pi]$, such that $\sin\theta = 1$. Thus, we conclude that for $\theta \in [0, 2\pi]$, $\sin\theta = 1$ if and only if $\theta = \pi/2$.

Finally, we take $\sin 3\pi/2$, which by (2.8) is equal to $-\sin(2\pi - 3\pi/2) = -\sin\pi/2 = -1$. Conversely, $\sin\theta = -1 \Leftrightarrow \sqrt{1 - \cos^2\theta} = -1$, which is impossible—meaning there is no $\theta \in [0, \pi]$, such that $\sin\theta = -1$. For interval $[\pi, 2\pi]$ however, we have $\sin\theta = -1 \Leftrightarrow -\sin(2\pi - \theta) = -1 \Leftrightarrow \sin(2\pi - \theta) = 1 \Leftrightarrow 2\pi - \theta = \pi/2 \Leftrightarrow \theta = 3\pi/2$. Thus, we conclude that for $\theta \in [0, 2\pi]$, $\sin\theta = -1$ if and only if $\theta = 3\pi/2$.

Graphs of sin **and** cos. Having the sine and cosine functions rigorously defined for all of $\mathbb{R}$ means we are able to plot them—see figure 4. (Anticipating the stipulated in remark 2.7, we now temporarily drop the practice of using Greek letters for the arguments of sin and cos—because labeling an abscissae axis θ, is more than a bit non-standard.)

One thing that might be suggested by this depiction, especially if one focuses on the domain interval $[-\pi, \pi]$, is that cosine appears to be an *even* function—i.e., $\cos x = \cos(-x)$—and sine appears to be an *odd* function—$\sin x = -\sin x$. This is indeed so, for any real x, and we prove it as follows. Write x as $y + 2\pi\lfloor x/2\pi \rfloor$, with $y \in [0, 2\pi[$ [8]— from whence, $\cos x = \cos y$ and $\sin x = \sin y$. Multiplying both sides by -1 yields $x = y + 2\pi\lfloor x/2\pi \rfloor \Leftrightarrow -x = -y - 2\pi\lfloor x/2\pi \rfloor$, with $-y \in]-2\pi, 0]$. Adding and subtracting 2π to the right hand side, we obtain $-x = (2\pi - y) + 2\pi(-\lfloor x/2\pi \rfloor - 1)$. As $2\pi - y \in [0, 2\pi[$, by (2.8), we have $\cos(-x) = \cos y$, and $\sin(-x) = -\sin y$—and it is now
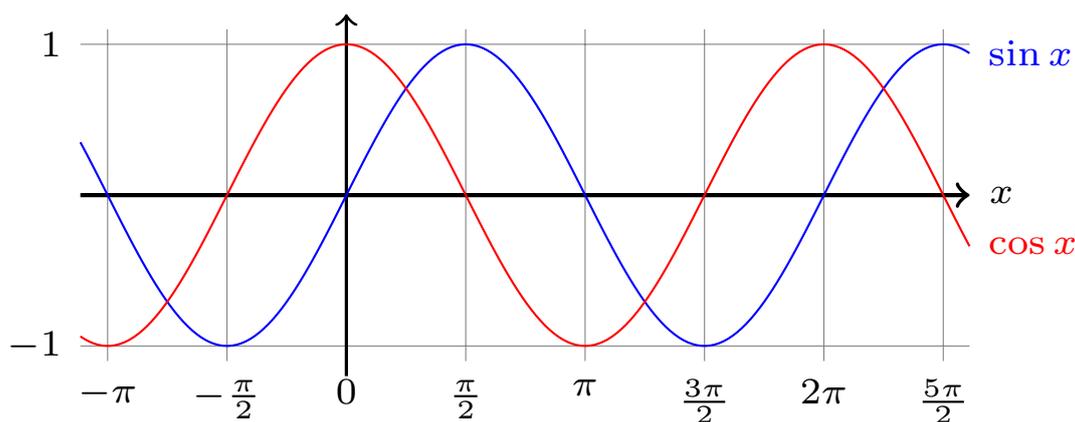
Figure 4: The sine and cosine functions.

straightforward that $\cos x = \cos(-x)$ and $\sin(-x) = -\sin x$.

But the most obvious thing suggested by the graphics of sine and cosine, is that the way in which we extended both functions to $\mathbb{R}$, yields *continuous* functions over all of $\mathbb{R}$. We next tackle the task of proving this rigorously.

Continuity of sin **and** cos**.** We have $\cos \theta = A^{-1}(\theta/2)$, where A^{-1} is a continuous bijection and $\theta \in [0, \pi]$. Hence, for $\alpha \in [0, \pi]$, we have: $\lim_{\theta \to \alpha} \cos \theta = \lim_{\theta \to \alpha} A^{-1}(\theta/2) = A^{-1}(\alpha/2) = \cos \alpha$.

The continuity of sine is now immediate, because given α as above, we have:

$$\lim_{\theta \to \alpha} \sin \theta = \lim_{\theta \to \alpha} \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \left(\lim_{\theta \to \alpha} \cos \theta \right)^2} = \sqrt{1 - \cos^2 \alpha}$$
$$= \sin \alpha$$

This shows that sine and cosine are continuous on $[0, \pi]$. From this and (2.8), it then follows that sin and cos are also continuous in $[\pi, 2\pi]$. Thus, in $[0, 2\pi]$ the only possible point of discontinuity is at $\theta = \pi$. We have $\lim_{\theta \to \pi^-} \cos \theta = \cos \pi = -1$, and $\lim_{\theta \to \pi^+} \cos \theta = \lim_{\theta \to \pi^+} \cos(2\pi - \theta) = \lim_{\theta \to \pi^-} \cos \theta = \cos \pi = -1$—thus the cosine function is continuous at $\theta = \pi$, and so it is continuous on $[0, 2\pi]$. For sine, the reasoning is analogous.

Similarly, when extending sin and cos from $[0, 2\pi]$ to $\mathbb{R}$, the only possible points of discontinuity would be at $\theta = 2\pi k$, with $k \in \mathbb{Z}$. Fix a value for k; to compute the lateral limits around $\theta = 2\pi k$, first note that on intervals $[2\pi(k-1), 2\pi k]$ and $[2\pi k, 2\pi(k+1)]$, both sin and cos take the same values as they do in $[0, 2\pi]$—this is an immediate consequence of (2.10). And so, it follows that $\lim_{\theta \to 2\pi k^-} \sin \theta = \lim_{\theta \to 2\pi^-} \sin \theta = \sin 2\pi = 0 = \sin 2\pi k$, and $\lim_{\theta \to 2\pi k^+} \sin \theta = \lim_{\theta \to 0^+} \sin \theta = \sin 0 =$

$0 = \sin 2\pi k$. For cosine, the reasoning is similar—allowing us to conclude that both sine and cosine are continuous over all of $\mathbb{R}$.

Derivatives of sin **and** cos. We now come to the final task in this section, computing the derivatives of sine and cosine; we begin with the latter. Again we have $\cos\theta = A^{-1}(\theta/2)$, where θ is restricted to $[0, \pi]$, and A^{-1} is a continuous bijection. Keeping in mind that $A'(x)$ is defined only for $x \in]{-}1, 1[$ (cf. the discussion surrounding (2.2)), and so the Derivative Of Inverse Rule can only be applied to $\theta \in]0, \pi[$, we use it, together with the Chain Rule, to obtain:

$$\cos'\theta = \left(A^{-1}(\theta/2)\right)' = (A^{-1})'(\theta/2) \cdot (1/2) = \frac{1}{A'\left[A^{-1}(\theta/2)\right]} \cdot \frac{1}{2}$$

$$= \frac{1}{A'(\cos\theta)} \cdot \frac{1}{2} = -\sqrt{1 - \cos^2\theta} = -\sin\theta$$

As for the derivative of the sine function, we have:

$$\sin'\theta = \left(\sqrt{1 - \cos^2\theta}\right)' = \frac{1}{2}\frac{-2\cos\theta\cos'\theta}{\sqrt{1 - \cos^2\theta}} = \frac{\cos\theta\sin\theta}{\sin\theta} = \cos\theta$$

Let us now extend these differentiation rules to values $\theta \in]\pi, 2\pi[$. Taking (2.8) into account, we have:

- For sin: $\sin'\theta = -\sin'(2\pi - \theta) = -\cos(2\pi - \theta) \times (-1) = \cos(2\pi - \theta) = \cos\theta$.
- For cos: $\cos'\theta = \cos'(2\pi - \theta) = -\sin(2\pi - \theta) \times (-1) = \sin(2\pi - \theta) = -\sin\theta$.

So, for $\theta \in]0, 2\pi[\setminus \{\pi\}$, we have $\sin' = \cos$ and $\cos' = -\sin'$. Now, for any $\theta \in \mathbb{R}$, except multiples of π, we saw above when discussing (2.10) that we can write $\theta = \gamma + 2\pi\lfloor\theta/2\pi\rfloor$, with $\gamma \in [0, 2\pi[$. As $\theta \neq k\pi$, for any integer k, this means $\gamma \in]0, 2\pi[\setminus \{\pi\}$. Hence, according to what we just proved, we have $\sin'\gamma = \cos\gamma$ and $\cos'\gamma = -\sin'\gamma$. Now let $t = 2\pi\lfloor\theta/2\pi\rfloor$. Similarly to what we observed above when discussing continuity, as t is a multiple of 2π, it is a direct consequence of (2.10) that cos takes the same value in $]t, 2\pi + t[\setminus \{\pi + t\}$ as it takes in $]0, 2\pi[\setminus \{\pi\}$—and the same is true for sin. Hence, if cos (resp. sin) is differentiable at a point $x \in]0, 2\pi[\setminus \{\pi\}$, then cos (resp. sin) is also differentiable at point $x + t \in]t, 2\pi + t[\setminus \{\pi + t\}$—and moreover $\cos'$ (resp. $\sin'$) has the same value at both points. But if $x = \gamma$, then $x + t$ is just θ, from whence we conclude that $\sin'\theta = \sin'\gamma = \cos\gamma = \cos\theta$ and $\cos'\theta = \cos'\gamma = -\sin\gamma = -\sin\theta$.

This just leaves the multiples of π, for which we require the following theorem:

Theorem 2.11 (Limit of the derivative). *Let X be a subset of $\mathbb{R}$ and a an interior point of X, and $f\colon X \to \mathbb{R}$ a function continuous in X and differentiable in $X \setminus \{a\}$, but such that $\lim_{x \to a} f'(x) = l$. Then $f'(a) = l$.*

Proof. See page 20 in appendix C.   ■

Now let X be a nonempty interval, such that the only multiple of π that is contained in X, is π itself. Consider the sine function: it is continuous in X, and for all $\theta \in X \setminus \{\pi\}$, we have $\sin' \theta = \cos \theta$. But cosine is continuous over all $\mathbb{R}$, which means in particular that $\lim_{\theta \to \pi} \cos \theta = \cos \pi = -1$. Theorem 2.11 now immediately gives $\sin' \pi = -1 = \cos \pi$. The same reasoning can be done for any multiple of π, which shows the sine function is differentiable over all $\mathbb{R}$—and its derivative is cosine. The reasoning is analogous to show that $\cos'(k\pi) = -\sin(k\pi)$ ($k \in \mathbb{Z}$).

Thus, we have shown that for any $\theta \in \mathbb{R}$, we have $\sin' = \cos$ and $\cos' = -\sin'$.

3   Taylor Series Of Sine And Cosine

In this section we derive the expressions of both functions in terms of the so-called Taylor series (see below). We leverage the fact that both sine cosine are solutions for the differential equation $f'' = -f$, $f\colon \mathbb{R} \to \mathbb{R}$—indeed, we would expect for any linear combination of sines and cosines to be a solution: $(a \cos x + b \sin x)'' = (-a \sin x + b \cos x)' = -(a \cos x + b \sin x)$. But this says nothing about whether any other solutions exist, so we proceed with a more generic approach. We shall require the values of cosine and sine at 0, which we have computed in the previous section: $\cos 0 = 1$ and $\sin 0 = 0$.

The first observation is that any solution f has derivatives of any order, over all of its domain: indeed, $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f'' = -f$, $f^{(3)} = -f'$, $f^{(4)} = -f'' = -(-f) = f = f^{(0)}$, and the cycle repeats indefinitely. Furthermore, f is continuous over all of $\mathbb{R}$, because its derivative is defined over all of $\mathbb{R}$. The same reasoning shows that $f^{(n)}$ is also continuous on $\mathbb{R}$, not just for $n = 0$ (which is f), but for any other $n \geq 1$ as well. In particular, f' is continuous on $\mathbb{R}$. Which entails that, if we take any $\varepsilon > 0$, and consider the interval $[-\varepsilon, \varepsilon]$, both f and f' are bounded on that interval.

We now enlist some "big guns" of mathematical analysis, namely Taylor polynomials with so-called Lagrange remainder. We have the following theorem.

Proposition 3.1. *Let $n \geq 0$ be an integer, X an interval of $\mathbb{R}$, and $f\colon X \to \mathbb{R}$ a function with continuous derivatives in X, up order $n+1$, and x_0 an interior*

point of X. Then for any $x \in X$, there exists c strictly between x_0 and x, such that:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \ldots$$
$$+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(x - x_0)^{n+1}$$

Proof. This is a restatement, in a different form, of Theorem 4 (Taylor's theorem) in Spivak [3, §19]—to which the reader is referred to for a proof. ∎

The series $\sum_{i=0}^{n} \left(f^{(i)}(x_0)(x - x_0)^i \right)/i!$ is called **Taylor polynomial of degree** n, and is usually denoted by $S_n(x)$. When $n \to +\infty$, we obtain the **Taylor series**. It is immediate that:

$$\frac{f^{(n+1)}(c)}{(n + 1)!}(x - x_0)^{n+1} = f(x) - S_n(x)$$

The left hand side is called the **Lagrange form of the Taylor remainder**. And we can now prove the result that we actually need.

Proposition 3.2. *If there exists $k \geq 0$ such that for any $n \geq 0$ we have $\left| f^{(n)}(x) \right| \leq k$ for all x in a neighbourhood[9] of x_0, then $f(x)$ equals its Taylor series for any x in that neighbourhood.*

Proof. See page 21 in appendix C. ∎

Now, returning to our reasoning where we left off, let $m = \max_{[-\varepsilon,\varepsilon]}|f(x)|$ and $m' = \max_{[-\varepsilon,\varepsilon]}|f'(x)|$. Thus for $M = \max\{m, m'\}$, we have $\left| f^{(n)}(x) \right| \leq M$, for all $n \geq 0$ and $x \in [-\varepsilon, \varepsilon]$. Hence $f^{(n)}$ is also bounded on the interval $]-\varepsilon, \varepsilon[$, which is a neighbourhood of $x = 0$, and making $k = M$, we can apply proposition 3.2 to conclude that any such function must coincide with its Taylor series on point $x = 0$. I.e., we must have:

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots$$

But because of the relation between derivatives of different orders we saw above, we can rewrite this as:

$$f(x) = f(0) + f'(0)x - \frac{f(0)}{2!}x^2 - \frac{f'(0)}{3!}x^3 + \frac{f(0)}{4!}x^4 + \frac{f'(0)}{5!}x^5 + \ldots$$

If we separately group terms with even and odd exponents, we obtain:

$$f(x) = \sum_{n=0}^{+\infty} \left((-1)^n f(0) \frac{x^{2n}}{(2n)!} + (-1)^n f'(0) \frac{x^{2n+1}}{(2n+1)!} \right)$$

We can further split the summation like this:

$$f(x) = f(0) \sum_{n=0}^{+\infty} \left((-1)^n \frac{x^{2n}}{(2n)!} \right) + f'(0) \sum_{n=0}^{+\infty} \left((-1)^n \frac{x^{2n+1}}{(2n+1)!} \right) \quad (3.3)$$

This equality holds for $x \in]-\varepsilon, \varepsilon[$, but as ε was arbitrarily chosen, it holds for any $x \in \mathbb{R}$. All of which shows that *any* function f that is a solution for $f'' = -f$, can be written in the form (3.3). But note the following: we know that one such solution is $f(x) = a \cos x + b \sin x$, where a, b are arbitrary values, and moreover, $f(0) = a$ and $f'(0) = b$. Contrasting this with (3.3), we conclude that the cosine function can be written as the left hand side summation, and that the sine function can be written as the right hand side summation.[10] From this it also follows that all solutions to $f'' = -f$ are of the form $f(x) = a \cos x + b \sin x$. The next result is now immediate.

Proposition 3.4. *The Taylor expansions of cosine and sine are:*

$$\cos x = \sum_{n=0}^{+\infty} \left((-1)^n \frac{x^{2n}}{(2n)!} \right) \quad and \quad \sin x = \sum_{n=0}^{+\infty} \left((-1)^n \frac{x^{2n+1}}{(2n+1)!} \right)$$

It is thus shown that the Taylor expansions of sine and cosine—often used, in analysis courses, as the *rigorous definitions* of sine and cosine—do indeed correspond to the "geometrical" sine and cosine one usually learns in high-school trigonometry.[11]

4 Some Properties of Sine And Cosine

We have $(\cos^2 x + \sin^2 x)' = 2 \sin x \cos x - 2 \cos x \sin x = 0$, meaning that $\cos^2 x + \sin^2 x$ is a constant function; and as $\cos^2 0 + \sin^2 0 = 1$, we conclude $\cos^2 x + \sin^2 x = 1$, for all $x \in \mathbb{R}$.

The well-known formulae for the sine and cosine of sums are also easy to derive. Letting c be a fixed value, we have $\sin'(x + c) = \cos(x + c)$ and $\cos'(x + c) = -\sin(x + c)$, which means $f(x) = \sin(x + c)$ is also a solution of $f'' = -f$—and hence, it can be written as $f(0) \cos x + f'(0) \sin x$, from where we conclude that for any $x, c \in \mathbb{R}$, we have:

$$\sin(x + c) = \sin x \cos c + \cos x \sin c$$

Differentiating both sides we obtain:

$$\cos(x + c) = \cos x \cos c - \sin x \sin c$$

Note that, even though they were deduced by fixing one of the parcels, these formulae apply to any two real numbers.

Recall that we computed in §2 that $\cos \pi = -1$ and $\sin \pi = 0$. Together with the above formulas for sines and cosines of sums, this yields:

$$\sin(x + \pi) = \sin x \cos \pi + \cos x \sin \pi = -\sin x$$
$$\cos(x + \pi) = \cos x \cos \pi - \sin x \sin \pi = -\cos x \tag{4.1}$$

This coincides with the intuition that arises from the "sine and cosine as Cartesian coordinates" approach depicted in figure 3. And this affords us another way to show that both sine and cosine have period 2π:[12]

$$\sin(x + 2\pi) = \sin((x + \pi) + \pi) = -\sin(x + \pi) = -(-\sin x) = \sin x$$
$$\cos(x + 2\pi) = \cos((x + \pi) + \pi) = -\cos(x + \pi) = -(-\cos x) = \cos x$$

But we can go beyond this, and show that 2π is the *smallest positive* (i.e., the *fundamental*) period. To do this, we rely on the fact that if T is the fundamental period of f, then for *all* other T' such that $f(x + T') = f(x)$, we have $T' = kT$, for some (possibly negative) integer k.[13] Now, we know that 2π is a period of sin and cos, thus the fundamental period, if different from 2π, must be a divisor of 2π—that is to say, it must be of the form $2\pi/n$, with $n > 1$ an integer. However, from (4.1), we know π is not a period of either function, and so we must actually have $n > 2$. Let us tackle sine first, by supposing we have $\sin(x + 2\pi/n) = \sin x$. But for $x = 0$, this would entail $\sin(2\pi/n) = \sin 0 = 0$, which we showed above—cf. §2—to be possible only for $n = 1$ or $n = 2$, and we have already established that we must have $n > 2$. Thus, no real of the form $2\pi/n$, with $n > 2$ an integer, can be a period of sin—which means 2π is its smallest period. Now for cosine, suppose we have $\cos(x + 2\pi/n) = \cos x$. Again for $x = 0$, this entails $\cos(2\pi/n) = \cos 0 = 1$, which we have also shown (ibid.) to be only possible if $n = 1$. Hence, no real of the form $2\pi/n$, with $n > 2$ an integer, can be a period of cos—which again means 2π is its smallest period.

5   Conclusion: A Discussion Of π

The main goal of this text—to establish the equivalence between geometric and analytical definitions of sine and cosine—has been accomplished. But I thought to end this writing with a note on π: in §2 we implicitly rely on the fact that for a circle of radius r, its perimeter is given by $2\pi r$, and its

area by πr^2. We then use this to determine the derivatives of sin and cos, and use these to, in §3, derive the Taylor polynomials for both functions. However, in the literature, when defining sin and cos in terms of Taylor polynomials, it is also customary to define π in terms of these functions.[14] That approach obviously does not work for us, because we have defined both cos and sin taking for granted that π was already defined.

Actually, that is not strictly true—what we relied on, was the fact that the ratio between the perimeter of a circle, and its diameter, is constant (i.e., it is a finite real number), for *any* circle. This cannot be proved—it has to be taken as an axiom of Euclidean geometry.[15] If we denote that constant finite real number by π, it is immediate that the perimeter of a circle of radius r is $2\pi r$. And furthermore, that its area is πr^2 can be shown by computing the following integral: $\int_0^r 2\pi t \, dt = 2\pi \int_0^r t \, dt = 2\pi(r^2/2 - 0) = \pi r^2$. Essentially, we "add up" the perimeters of all inner circles, for all radii from 0 to r. This suffices for the reasoning carried out in §2—but it does not help one to compute the actual value of π, because we haven't really defined it.

So, how to actually define π? There are many possibilities, but one that relates to the work done in §2, is to define π as twice the area under the upper unit circumference—which can be computed as $\int_{-1}^{1} \sqrt{1 - t^2} dt$. That is, we set $\pi \overset{\text{def}}{=} 2 \int_{-1}^{1} \sqrt{1 - t^2} dt$.

A   Sine, cosine and triangles

Take any right triangle: it is well-known from Euclidean geometry that we can always obtain an equivalent triangle, by dividing the length of all sides by the length of the hypotenuse, thus obtaining a similar triangle with equal angles to the original one, and a length 1 hypotenuse. Let this new triangle be the one depicted in figure 5a. It can be inscribed in the trigonometric circle, by placing the hypotenuse (line segment $\overline{AB}$) "on top" of line segment $\overline{OP}$ in figures 1 and 2. There are two ways of accomplishing this. One is to have line segment $\overline{AC}$ lay on top of the positive abscissae semi-axis (having point A coincide with point O of figures 1 and 2): from which we immediately conclude that $\cos\alpha = \overline{AC}$ and $\sin\alpha = \overline{BC}$. The other is to have line segment $\overline{BC}$ lay on top of the positive abscissae semi-axis (having point B coincide with point O of figures 1 and 2): from which we immediately conclude that $\cos\beta = \overline{BC}$ and $\sin\beta = \overline{AC}$. This shows that our functions of sine and cosine also apply to (right) triangles: indeed, we can look at the trigonometric circle as a *generalization* of right-triangle trigonometry, where one learns that cosine (resp. sine) of an angle is the ratio between the length of the side adjacent (resp. opposite) to that angle, and the hypotenuse.
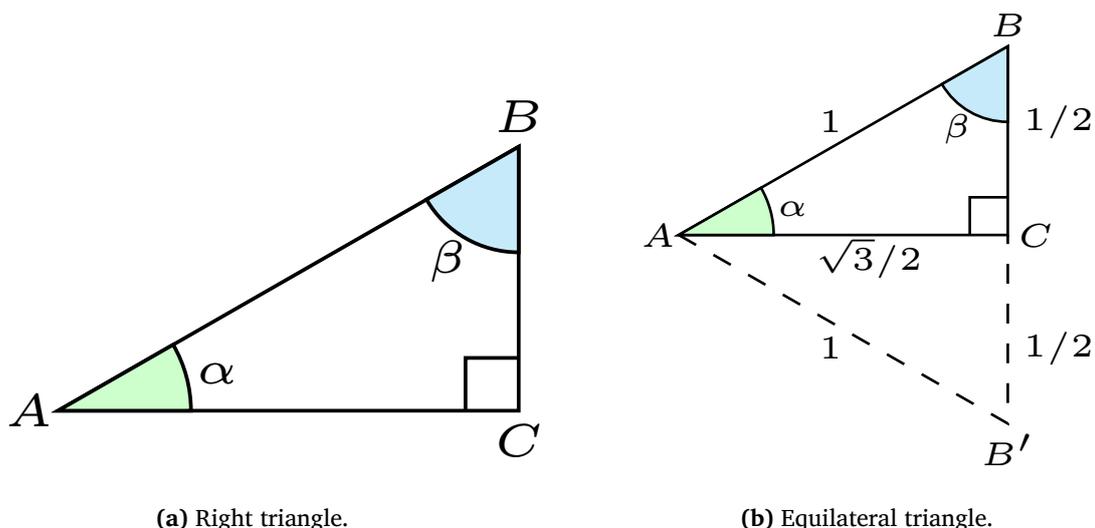
(a) Right triangle.   (b) Equilateral triangle.

Figure 5: Triangles and trigonometry.

However, this "particularization" of trigonometry—i.e., going from the trigonometric circle to right-triangles—does allow us to compute the cosine and sine for some more concrete values, in addition to what was done in §2. For example, if $\overline{AC}$ and $\overline{BC}$ have the same length, then $\alpha = \beta$, and as all the internal angles of any triangle always add to π, we must have $\alpha = \beta = \pi/4$. Furthermore, as $\overline{AB}$ has length 1, by Pythagoras' theorem, the length of both $\overline{AC}$ and $\overline{BC}$ must be $\sqrt{2}/2$. Which means that $\cos \pi/4 = \sin \pi/4 = \sqrt{2}/2$.

But in figure 5a, the sides $\overline{AC}$ and $\overline{BC}$ do *not* have the same length—in fact, the length of $\overline{BC}$ is *half* of that of $\overline{AC}$. This is to allow us to build an *equilateral* triangle, by "doubling" our existing triangle—cf. figure 5b.

Now, as the all the sides of the equilateral triangle ABB' have the same length, so will all the angles be equal—and as the total must sum to π, this means that each angle will have an amplitude of $\pi/3$ radians; in particular, we will have $\beta = \pi/3$. But as the length of $\overline{B'C}$ is equal to that of $\overline{BC}$, this means the amplitude of $C\hat{A}B'$ is α—or equivalently, the amplitude $B\hat{A}B'$ is 2α. As $B\hat{A}B' = \pi/3$, $\alpha = \pi/6$. As for side length, if $\overline{AB}$ has length 1, and $\overline{BC}$ has half of that $(1/2)$, then by Pythagoras' theorem, the length of $\overline{AC}$ is $\sqrt{3}/2$. It is now immediate that $\sin \pi/6 = \cos \pi/3 = 1/2$ and $\sin \pi/3 = \cos \pi/6 = \sqrt{3}/2$.

B   Generalities on periodic functions

A function $f \colon \mathbb{R} \to \mathbb{R}$ is said to be *periodic*, with period T, if there exists a positive real T such that, for any x in the domain of f, we have $f(x+T) = f(x)$. The period is not unique: it is easy to see that for every integer n (including negatives) we have $f(x+nT) = f(x)$. Indeed, letting $x = x+T$ in $f(x+T) = f(x)$, yields $f(x+T+T) = f(x+T) \Leftrightarrow f(x+2T) = f(x+T)$,

from whence $f(x + 2T) = f(x)$. A simple induction argument then shows that we have $f(x + nT) = f(x)$, for any integer $n \geq 1$ (the case $n = 0$ is trivial). Now if set $y = x + nT$, this means $x = y - nT$, and thus replacing above, we obtain $f(y - nT) = f(y)$. As x is arbitrary, so is y, establishing that if $f(x + T) = f(x)$, then also $f(x + nT) = f(x)$, for any integer n.[16]

Conversely, if T is the smallest (positive) period of f—i.e., T is the *fundamental period*—then for *all* other T' such that $f(x + T') = f(x)$, we have $T' = kT$, for some (possibly negative) integer k. To see this, suppose that there existed a T' such that $f(x + T') = f(x)$ and T' was not an integer multiple of T—and let us first suppose T' is positive. As by assumption T' is not the fundamental period, this means $T' > T$. If we set $n = \lfloor T'/T \rfloor$, then nT is such that $0 < T' - nT < T$. We showed above that nT is also a period of f, and hence we would have $f(x + nT) = f(x)$ and $f(x + T') = f(x)$, which of course means $f(x + nT) = f(x + T')$. But $T' = nT + (T' - nT)$, which means $f(x + nT) = f(x + nT + (T' - nT))$, or writing $y = x + nT$, $f(y) = f(y + (T' - nT))$—and as $T' - nT$ is non-negative, it is also a period of f. But $T' - nT$ is smaller than T, which contradicts the fact that T is the smallest period of f. Thus T' cannot be positive—so now let it be negative. If we had $T' > -T$, this would mean $-T' < T$, and if T' is negative, then $-T'$ is positive, which would mean that T was not the fundamental period, which is absurd. So we must have $T' < -T \Leftrightarrow -T' > T$. As shown in the previous paragraph, from $f(x + T') = f(x)$ follows that $f(x + (-T')) = f(x)$, i.e., $-T'$ is a period of f, greater than T. Moreover, if T' is not an integer multiple of T, then neither is $-T'$. But by the same reasoning as above, we reach the conclusion that $-T' - \lfloor -T'/T \rfloor T$ must also be a period of f. Which again contradicts the fact that T is the fundamental period, because $0 < -T' - \lfloor -T'/T \rfloor T < T$. We thus conclude that if $f(x + T') = f(x)$ holds, then T' is an integer multiple of the fundamental period T.

The only question left unanswered is whether any periodic function always has a fundamental period. And the answer is **no**, as the following example—the so-called Dirichelet function—proves:

$$1_{\mathbb{Q}} = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

The reasons for denoting this function by $1_{\mathbb{Q}}$ are unimportant for our purposes.[17] It has the property that *any* positive rational is a period, which follows from the fact that given any rational r, $r + \alpha$ is rational if and only if α is rational. Thus, α and $r + \alpha$ are either both rational, or both irrational—and in either case, $1_{\mathbb{Q}}(\alpha) = 1_{\mathbb{Q}}(r + \alpha)$.

However, *for continuous functions*, it does hold that any periodic function, *if it is not a constant function*, it has a fundamental period.

Proposition B.1. *Let f be a continuous and periodic function. Then, it has a fundamental period if and only if it is not constant.*

Proof. We will show that f is constant if and only if it has no fundamental period (i.e., if we can find periods arbitrarily close to zero). ($\to$) If f is a constant function, then it is obvious that it is periodic, and that there does not exist a smallest period. ($\leftarrow$) If f has no fundamental period, but is nonetheless a periodic function, then that means we can find a strictly decreasing sequence of positive terms, $\{T_n\}$, such that $T_n \to 0$. Now let a, b be two distinct points of the domain of f. For any n, we can write $b - a = T_n \lfloor (b-a)/T_n \rfloor + r_n$, with $0 \le r_n < T_n$ (note this holds regardless of whether $b - a$ is positive or negative). Rearranging, we have $b - r_n = a + T_n \lfloor (b-a)/T_n \rfloor$. As $0 \le r_n < T_n$ and $T_n \to 0$, by the squeeze test[18] we have $r_n \to 0$. Now, on the one hand, as every T_n is a period of f, we have $f(a + T_n \lfloor (b-a)/T_n \rfloor) = f(a)$, which is to say that all the terms of the sequence $f(b - r_n)$ are equal to $f(a)$. But on the other hand, $\lim_{n \to \infty} b - r_n = b$, and as f is continuous, by lemma C.1 we have $\lim_{n \to \infty} f(b - r_n) = f(\lim_{n \to \infty} b - r_n) = f(b)$. Hence, we conclude that $f(b) = f(a)$—and as a and b were arbitrary points of the domain of f, we conclude that f is a constant function.   ∎

We finish with the following remark: it is straightforward to observe that if we have a sequence $T_n \to T$, such that for any n, we have $f(x + T_n) = f(x)$, where f is a continuous function, and x is any value in its domain, then we must also have $f(x + T) = f(x)$. To see why this is, let x be fixed; it is immediate that the sequence $x + T_n$ converges to $x + T$. Now, on the one hand, the sequence $f(x + T_n)$ is constant—all its terms are equal to $f(x)$. On the other, because f is continuous, again via lemma C.1, we have $\lim_{n \to \infty} f(x + T_n) = f(\lim_{n \to \infty} x + T_n) = f(x + T)$. Thus we conclude that $f(x + T) = f(x)$. Observe that for this result, neither the terms T_n, nor T need be positive. But if $T = 0$, we are left with a rather trivial condition, that is true for any function, namely that $f(x + 0) = f(x)$. Proposition B.1 also clarifies what happens when $T = 0$ and the terms T_n are nonzero (if the terms are zero, then also $T = 0$, and again, the result is trivial).[19]

C   Proofs

The following lemma is needed for proving theorem 2.3.

Lemma C.1. *Let X be a subset of $\mathbb{R}$ and $f\colon X \to \mathbb{R}$ be a function. The function f is continuous at point $a \in X$ if and only if for every sequence $\{x_n\}$ of points of X that converges to a, we have $f(x_n) \to f(a)$.*

Proof. Let $\{x_n\}$ be a sequence of points of X such that $\{x_n\} \to a$. By definition of convergent sequence, this means that ($\varepsilon \in \mathbb{R}$ and $n, p \in \mathbb{N}$):

$$\forall \varepsilon > 0 \; \exists p \; \forall n \;\; n \geq p \Rightarrow |x_n - a| < \varepsilon \tag{C.2}$$

($\to$) We assume f is continuous at a, and want to show that this implies $\{f(x_n)\} \to f(a)$. By definition of continuity, the following holds:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in X \;\; 0 < |x - a| < \delta \Rightarrow |f(x) - a| < \varepsilon \tag{C.3}$$

The condition we want to show—$\{f(x_n)\} \to f(a)$—translates too:

$$\forall \varepsilon > 0 \; \exists p \; \forall n \;\; n \geq p \Rightarrow |f(x_n) - f(a)| < \varepsilon \tag{C.4}$$

The first observation, is that for any n such that $x_n = a$, (C.4) holds trivially, so we can assume $x_n \neq a$. For these, it follows from (C.3) that given any real $\varepsilon > 0$, there exists a real $\delta > 0$ such that $\forall x \in X, \; 0 < |x_n - a| < \delta \Rightarrow |f(x_n) - f(a)| < \varepsilon$. And from (C.2) we know that for *that* δ, there exists p such that $n \geq p \Rightarrow |x_n - a| < \delta$. Combining these two statements yields (C.4).

($\leftarrow$) We assume that $\{x_n\} \to a$ implies $\{f(x_n)\} \to f(a)$, for any arbitrary sequence $\{x_n\}$ of points of X. We want to show that this in turn implies that f is continuous at a, i.e., that $\lim_{x \to a} f(x) = f(a)$. We will prove this via the contrapositive, that is, we will assume that f is *not* continuous at a, and show that in such a case, there exists a sequence $\{x_n\}$ such that $\{x_n\} \to a$ and $\{f(x_n)\} \not\to f(a)$.

So, if f is not continuous at a, this means that the following condition holds:

$$\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists x \in X \;\; 0 < |x - a| < \delta \wedge |f(x) - a| \geq \varepsilon \tag{C.5}$$

Fix an ε for which (C.5) holds. Then, if we set $\delta = 1$, there exists at least one value of x such that the sub-condition $0 < |x - a| < \delta \wedge |f(x) - a| \geq \varepsilon$ holds—let x_1 equal that value of x. More generally, let x_n be a value of x for which the same sub-condition holds when $\delta = 1/n$. It is clear that $\{x_n\} \to a$, but $\{f(x_n)\} \not\to f(a)$, which concludes the proof. $\blacksquare$

Theorem 2.3 (Continuity of the inverse). *Let $a, b \in \mathbb{R}$ be two reals, with $a < b$, and let $f \colon [a, b] \to \mathbb{R}$ be a continuous and strictly decreasing function. Then f is a bijection from $[a, b]$ to $[f(b), f(a)]$, and its inverse, f^{-1} is also strictly decreasing, and continuous.*

Proof. It is immediate that f being strictly monotone (decreasing in the present case), it is also injective. We must show that for any $x \in [a, b]$,

$f(x) \in [f(b), f(a)]$. This is trivial for either $x = a$ or $x = b$; for $x \in$ $]a, b[$, f being strictly decreasing means that $a < x < b$ implies $f(a) > f(x) > f(b)$. From here it also follows that $[f(b), f(a)]$ is a nonempty interval.

To show that f is surjective in $[f(b), f(a)]$, we first note that as it is continuous in $[a, b]$, by the Intermediate Value Theorem, for any $y \in [f(b), f(a)]$, there exists $x \in [a, b]$ such that $f(x) = y$—and this x is unique due to f being injective. Hence, f is bijective, and thus we can define $f^{-1} \colon [f(b), f(a)] \to [a, b]$.

To show that f^{-1} is also strictly decreasing, let $y_1, y_2 \in [f(b), f(a)]$, with $y_1 < y_2$, and let $x_1 = f^{-1}(y_1) \Leftrightarrow y_1 = f(x_1)$ and $x_2 = f^{-1}(y_2) \Leftrightarrow y_2 = f(x_2)$. Reasoning by contradiction, assume that $f^{-1}(y_1) \leq f^{-1}(y_2)$. We have: $x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2) \Rightarrow y_1 \geq y_2$, which is a contradiction.[20] Thus we conclude that $f^{-1}(y_1) > f^{-1}(y_2)$.

Our last task is to show that f^{-1} is continuous—and for this, we shall use lemma C.1 to show that for any sequence $\{y_n\}$ in $[f(b), f(a)]$ such that $\{y_n\} \to y_0$, we have $\{f^{-1}(y_n)\} \to f^{-1}(y_0)$. As f is bijective, for any y_n there exists an unique x_n such that $y_n = f(x_n)$—namely, $x_n = f^{-1}(y_n)$. In particular, $x_0 = f^{-1}(y_0) \Leftrightarrow y_0 = f(x_0)$. Thus we can rewrite $\{y_n\} \to y_0$ as $\{f(x_n)\} \to f(x_0)$. Due to the continuity of f, **if $\{x_n\}$ is convergent**, it must converge to x_0—otherwise (i.e., if it converged to any other value, say $c \neq x_0$), by lemma C.1 and the unicity of limits we would have $f(x_0) = f(c)$, which is impossible as f is injective.[21] To prove that $\{x_n\}$ is indeed convergent, we first obverse that for any n, $x_n \in [a, b]$, which is a *bounded* subset of $\mathbb{R}$. Hence, if $\{x_n\}$ is divergent, as neither it nor any of its subsequences can diverge to (positive or negative) infinity, it must be because there exist at least two subsequences that converge to different limits.[22] Let us then, suppose there exist two subsequences of $\{x_n\}$, $\{x_{n_i}\}$ and $\{x_{n_j}\}$, such that $\{x_{n_i}\} \to c_i$ and $\{x_{n_j}\} \to c_j$, with $c_i \neq c_j$. As f is continuous, $\{f(x_{n_i})\} \to f(c_i)$ and $\{f(x_{n_j})\} \to f(c_j)$. However, as both $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are subsequences of $\{x_n\}$, $\{f(x_{n_i})\}$ and $\{f(x_{n_j})\}$ are subsequences of $\{f(x_n)\}$, which converges to $f(x_0)$—and thus, so must $\{f(x_{n_i})\}$ and $\{f(x_{n_j})\}$. This entails $f(c_i) = f(c_j) = f(x_0)$, and f being injective, we conclude $c_i = c_j$, which is a contradiction. Hence, $\{x_n\} \to x_0$. We now have $\{f^{-1}(y_n)\} = \{f^{-1}[f(x_n)]\} = \{x_n\} \to x_0 = f^{-1}(y_0)$, i.e., $\{f^{-1}(y_n)\} \to f^{-1}(y_0)$. As $\{y_n\}$ was an arbitrary convergent sequence, by lemma C.1 we conclude that f^{-1} is continuous on $[f(b), f(a)]$. ∎

Remark C.6. An analogous result also holds when f is strictly increasing—in which case, f^{-1} will also be strictly increasing. △

Theorem 2.11 (Limit of the derivative). *Let X be a subset of $\mathbb{R}$ and a an interior point of X, and $f \colon X \to \mathbb{R}$ a function continuous in X and differentiable in $X \setminus \{a\}$, but such that $\lim_{x \to a} f'(x) = l$. Then $f'(a) = l$.*

Proof. We want to prove that:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in X \setminus \{a\} \quad 0 < |x - a| < \delta \Rightarrow \left| \frac{f(x) - f(a)}{x - a} - l \right| < \varepsilon$$

$$\text{(C.7)}$$

Applying Lagrange's theorem to f in the closed interval with extrema a and x, we know that there exists c (dependent on x) such that:

$$\left| \frac{f(x) - f(a)}{x - a} - l \right| = |f'(c) - l| \qquad \text{(C.8)}$$

Now $\lim_{x \to a} f'(x) = l$ means that:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in X \setminus \{a\} \quad 0 < |x - a| < \delta \Rightarrow |f'(x) - l| < \varepsilon \quad \text{(C.9)}$$

Thus, given any $\varepsilon > 0$, we chose $\delta > 0$ so as to verify the implication in (C.9). For any x such that $0 < |x - a| < \delta$, Lagrange's theorem also tells us that the c that corresponds to that x according to (C.8), belongs to the open interval with extrema a and x. This means that $0 < |c - a| < \delta$—from which we have $|f'(c) - l| < \varepsilon$. In other words, for any x such that $0 < |x - a| < \delta$, we have $\left| (f(x) - f(a))/(x - a) - l \right| < \varepsilon$. It is now immediate that (C.7) holds.

∎

Proposition 3.2. *If there exists $k \geq 0$ such that for any $n \geq 0$ we have $\left| f^{(n)}(x) \right| \leq k$ for all x in a neighbourhood of x_0, then $f(x)$ equals its Taylor series for any x in that neighbourhood.*

Proof. Let X be a neighbourhood of x_0 verifying the required conditions, and let $x \in X$. Apply proposition 3.1 to f, and set

$$S_n(x) = \sum_{i=0}^{n} \left(f^{(i)}(x_0)(x - x_0)^i \right)/i!$$

We have:

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} = f(x) - S_n(x)$$

for some c strictly between x and x_0—and thus, $c \in X$. We have:

$$|f(x) - S_n(x)| = \left| f^{(n+1)}(c) \right| \cdot \frac{\left| (x - x_0)^{n+1} \right|}{(n+1)!} \leq k \frac{|(x - x_0)|^{n+1}}{(n+1)!}$$

Now by the ratio test,[23] one sees that $\sum y^n/n!$ is convergent, for any $y \in \mathbb{R}$—and so, $y^n/n! \to 0$ when $n \to +\infty$. Hence, $k(|(x - x_0)|^{n+1})/((n+1)!) \to 0$, entailing $f(x) - S_n(x) \to 0$ for every x in a neighbourhood of x_0, and thus, f equals its Taylor series in that neighbourhood. ∎

Notes

1. For an example of this approach, see Tao [4, §4.7]. Being defined for any value in $\mathbb{C}$, they are of course defined for all of $\mathbb{R}$.

2. Cf. Tao, loc. cit.

3. As is conventional, counter-clockwise angles are considered **positive**, and clockwise angles are considered **negative**.

4. Recall that if $u(x)$ is a function, then $\left(\sqrt{u(x)}\right)' = u'(x)/(2\sqrt{u(x)})$.

5. See, e.g., Tao [5, Prop. 10.1.10, in §10].

6. Recall that $\lfloor x \rfloor$ is the largest integer not greater than the real x. To show uniqueness, suppose there exist θ, θ' and k, k' such that $\theta + 2\pi k = \theta' + 2\pi k'$. This is the same as $\theta - \theta' = 2\pi(k' - k)$. But as both θ and θ' are in $[0, 2\pi[$, $\theta - \theta' \in]-2\pi, 2\pi[$—and the only multiple of 2π in that interval is 0. Hence, $k' - k = 0 \Leftrightarrow k' = k$, which of course entails $\theta = \theta'$.

7. Some of the "converse" results below, are needed in §4, when discussing the periodicity of sine and cosine.

8. Cf. the discussion after (2.10), if needed.

9. A **neighbourhood** of a point a is an open interval of the form $]a - \varepsilon, a + \varepsilon[$, for some real number $\varepsilon > 0$.

10. It is a simple exercise to apply the ratio test—see for example, Tao [5, §7.5]—to show that both series have an infinite radius of convergence, meaning they converge for any $x \in \mathbb{R}$—just as we would expect, since our sine and cosine functions are also defined all over $\mathbb{R}$.

11. As a "sanity check," it is immediate to verify that according to their respective Taylor expansions, we have $\cos 0 = 1$ and $\sin 0 = 0$, just as computed in §2. Also, power series—of which the Taylor series is a particular case—are indefinitely differentiable, and so term-wise differentiation shows that $\sin' x = \cos x$ and $\cos' x = -\sin x$—which again coincides with our previous finding in §2.

12. Recall that by construction, we already expect for 2π to be a period of both sine and cosine. Cf. (2.10).

13. In appendix B, it is shown that every continuous periodic and non-constant function, *has* a smallest period. The previous assertion about periodic functions is also proved therein.

14. Tao [4, §4.7], for example, defines π as the smallest zero of the sin function in $]0, +\infty[$.

15. In fact, in other, so-called *non-Euclidean* geometries, that ratio can be different for different circles! For example, in spherical geometry, you can have 'π'= 2! See the answer of user SRM (April 21$^{\text{st}}$, 2014), on this `Math StackExchange` thread [1].

16. But note that only the positive multiples of T are periods of f, because by definition, a period has to be positive.

17. Nevertheless, for the curious reader, the explanation is as follows: the Dirichelet function is what is called the *indicator function* for the set of rational numbers, $\mathbb{Q}$. Generically, given a set X, its indicator function, denoted 1_X, is a function that evaluates to 1 for any element that belongs to X, and 0 for any element that doesn't.

18. Tao [5, Corollary 6.4.14 in §6.4], for example.

19. Proposition B.1 requires that $\{T_n\}$ be a strictly decreasing sequence of positive terms; however, given a sequence $\{T'_n\}$ of nonzero terms that converges to zero, we can construct a strictly decreasing sequence $\{T_n\}$, having only positive terms, that also converges to zero. Here is how: let $T_1 = T'_1$, if T'_1 is positive, and $-T'_1$ if it is negative. Let T'_i be the next term of sequence T'_n verifying $|T'_i| < |T'_1|$ (one such term can always be found, because $T'_n \to 0$). Set $T_2 = T'_i$, if T'_i is positive, and $-T'_i$ if it is negative. And so on, and so forth…

20. The step $x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$ might need additional justification. If $x_1 = x_2$, then of course we have $f(x_1) = f(x_2)$. And as f is strictly decreasing, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$. Hence, $x_1 \leq x_2$ implies either $f(x_1) = f(x_2)$ or $f(x_1) > f(x_2)$, i.e., implies $f(x_1) \geq f(x_2)$.

21. Note that as all the terms of $\{x_n\}$ are in $[a, b]$, so must the putative limit c.

22. Recall that by the Bolzano-Weierstraß theorem any bounded sequence has a convergent subsequence.

23. Cf. note 10 above.

References

1. MATH STACKEXCHANGE, 2014. *Is there any geometry where ratio of circles's circumference to its diameter is rational?*. See https://math.stackexchange.com/questions/762423/is-there-any-geo metry-where-ratio-of-circles-circumference-to-its-diameter-is-r, last access on July 30, 2023. Cited on p. 22.

2. **Sarrico**, Carlos, 1997. *Análise Matemática*. Lisboa: Gradiva. In Portuguese. Cited on p. 2.

3. **Spivak**, Michael, 1976. *Calculus*. W. A. Benjamin, Inc. Cited on pp. 2 and 12.

4. **Tao**, Terence, 2022 [2006]. *Analysis II*. New Delhi: Hindustan Book Agency, 4th edition. Cited on p. 22.

5. **Tao**, Terence, 2022 [2006]. *Analysis I*. New Delhi: Hindustan Book Agency, 4th edition. Cited on p. 22.